Uniqueness of Reduced Row Echelon Form

Many introductory linear algebra books either fail to mention this result, omit its proof, or present a proof which is unnecessarily complicated or uses arguments beyond the context in which the result occurs. Here's a proof which, hopefully, suffers from none of these deficiencies.

Theorem: The reduced (row echelon) form of a matrix is unique.

Proof (W.H. Holzmann): If a matrix reduces to two reduced matrices R and S, then we need to show $R = S$. Suppose $R \neq S$ to the contrary. Then select the first (leftmost) column at which R and S differ and also select all leading 1 columns to the left of this column, giving rise to two matrices R' and S'. For example, if

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} 1 & 2 & 0 & 7 & 9 \\ 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

then

$$R' = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad S' = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{bmatrix}.$$

In general,

$$R' = \left[\begin{array}{c|c} I_n & \mathbf{r}' \\ \hline O & \mathbf{0} \end{array}\right] \qquad \text{or} \qquad \left[\begin{array}{c|c} I_n & \mathbf{0} \\ \hline & 1 \\ O & 0 \\ & \vdots \end{array}\right],$$

and

$$S' = \left[\begin{array}{c|c} I_n & \mathbf{s}' \\ \hline O & \mathbf{0} \end{array}\right] \qquad \text{or} \qquad \left[\begin{array}{c|c} I_n & \mathbf{0} \\ \hline & 1 \\ O & 0 \\ & \vdots \end{array}\right].$$

It follows that R' and S' are (row) equivalent since deletion of columns does not affect row equivalence, and that they are reduced but not equal.

Now interpret these matrices as augmented matrices. The system for R' has a *unique* solution $\mathbf{r}'$ or is inconsistent, respectively. Similarly, the system for S' has a *unique* solution $\mathbf{s}'$ or is inconsistent, respectively. Since the systems are equivalent, $\mathbf{r}' = \mathbf{s}'$ or both systems are inconsistent. Either way $R' = S'$, a contradiction. ∎