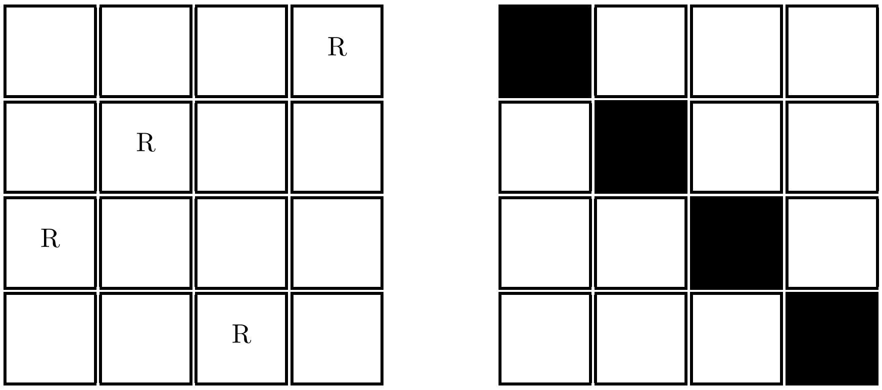
Permutations with Restricted Positions and Rook Polynomials

A permutation of 1, 2, …, n corresponds to a placing of n mutually noncapturing rooks on an $n \times n$ board, or equivalently, is a matrix of 0's and 1's with exactly one 1 in each row and each column. For example, the left figure illustrates the permutation 3 2 4 1, where we put a rook R in the 1st row and 4th column because the 1 is in the 4th position of the permutation, etc.:



In a permutation, such as a derangement, we can indicate restricted, that is, forbidden positions by shading appropriate squares on the board. Any kind of permutation with restricted positions can be indicated this way. For example, the restricted positions D_4 for a 4-derangement are illustrated on the right above.

Let C denote a board with shadings. In inclusion-exclusion, we would consider the set A_i, the set of all permutations in which rook i **is** in a forbidden position. As well we would consider multiple intersections of these kinds of sets, which leads us to consider the following.

Let $r_k(C)$ be the *number of arrangements of k rooks such that all k are in forbidden positions.*

If we let the remaining $n-k$ rooks occupy any other mutually noncapturing positions (there are $(n-k)!$ ways) then $r_k(C)(n-k)!$ is precisely the S_k of the inclusion-exclusion formula. Thus by inclusion-exclusion:
The number of permutations with no objects in forbidden positions is

$$r_0(C)n! - r_1(C)(n-1)! + r_2(C)(n-2)! + \cdots$$

The polynomial $r(C) = r_0(C) + r_1(C)x + r_2(C)x^2 + \cdots + r_n(C)x^n$ is called the **rook polynomial** of C. Note $r_0(C) = 1$ always.

The main advantage of this approach is that $r(C)$ can be computed relatively easily for a given pattern because there are a number of reductions available.

(I) If

$$C = \begin{matrix} C_1 & E_1 \\ E_2 & C_2 \end{matrix}$$

where E_1 and E_2 are blank (no shaded squares) then $r(C) = r(C_1) \cdot r(C_2)$.
This is essentially the multiplication rule as follows. Place i rooks in C_1 and $k-i$ rooks in C_2. There are $r_i(C_1) \cdot r_{k-i}(C_2)$ ways. Then add up cases — add over all the i's.

(II) By a direct counting argument it can be shown that

$$r(F_j) = 1 + \binom{j}{1} jx + \binom{j}{2} j(j-1)x^2 + \cdots + \binom{j}{j} j!x^j$$

where F_j is the size j board with all squares shaded. Namely, choose the rows for the rooks, and then j represents the columns available for the first rook, $j-1$ columns for second rook, etc.

(III) Let C be a board in which square q is shaded. Let C_S be the subboard obtained from C by removing the row and column of q and let C_e be C but with square q unshaded. Then

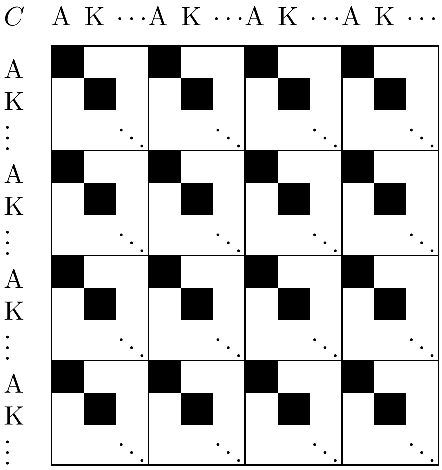
$$r(C) = r(C_e) + r(C_S)x$$

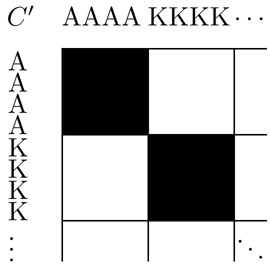
since in the case where we do not put a rook on q we might as well remove q giving C_e and in the case where we do put a rook on q we have one fewer rook to give away — the reason for the x — and the remaining rooks must go on C_S.

If E is empty then $r(E) = 1$. Also $r(\blacksquare) = 1 + x$. Applying (I) gives $r(D_j) = (1 + x)^j$ where D_j is the j-derangement board, that is, the size j board with just the main diagonal squares shaded.

EXAMPLE: Flip the 52 cards from a standard deck in succession while calling out the 13 ranks Ace, King, Queen, etc. in order, and restart at the Ace after you call out the rank two. What is the probability that you will never flip over a card with the called rank?

This is not a 52-derangement, although it is related. The board C for this situation consists of a four by four array of D_{13}'s (see left figure). It can be rearranged into a more tractable board C' (see right figure) as follows: Put all the Aces together, all the Kings together, etc. This is equivalent to calling out Ace four times, then King four times, etc. The board C' is 13 copies of F_4 down the diagonal.





By (I) 13 times and (II) with $j = 4$ the rook polynomial is

$$(1 + 16x + 72x^2 + 96x^3 + 24x^4)^{13}$$

Expanding this out and replacing each x^i by $(-1)^i(52 - i)!$ gives the number of arrangements that do not have forbidden positions. Using the computer algebra system MAPLE, this number was computed to be

$$1309302175551177162931045000259922525308763433362019257020678406144.$$

Dividing this number, 1.309×10^{66}, by 52! which is about 8.065×10^{67} gives the probability to be approximately .0162327. In summary, you have about a one and a half percent chance of getting through the deck without a match.