Approximate Integration and Simpson's Rule

To approximate

$$\int_a^b f(x)\,dx,$$

let $[a,b]$ be subdivided into n subintervals each of length $h_n = \dfrac{b-a}{n}$.
The subdivision points are $x_0 = a,\, x_1 = a+h,\, \ldots,\, x_j = a+jh,\, \ldots,\, x_n = b$ with values $y_j = f(x_j)$.

1. Trapezoid Rule

$$\mathcal{R}_{0,n} = T_n = h_n\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right)$$

This is obtained by repeating the area of the trapezoid (degree 1 polynomial) approximation $h_n\left(\frac{1}{2}y_0 + \frac{1}{2}y_1\right)$ over successive subintervals. It is an exact approximation for linear $f(x)$ (degree 1 polynomials).

2. Simpson's Rule

$$\mathcal{R}_{1,n} = S_n = h_n\left(\frac{1}{3}y_0 + \frac{4}{3}y_1 + \frac{2}{3}y_2 + \frac{4}{3}y_3 + \cdots + \frac{2}{3}y_{n-2} + \frac{4}{3}y_{n-1} + \frac{1}{3}y_n\right),$$ where n is **even**.

This is obtained by repeating the area of the quadratic (degree 2 polynomials) approximation $h_n\left(\frac{1}{3}y_0 + \frac{4}{3}y_1 + \frac{1}{3}y_2\right)$ over successive pairs of subintervals. It is an exact approximation for cubics (degree 3 polynomials).
Note

$$\begin{aligned}
S_n &= h_n\left(\frac{2}{3}y_0 + \frac{4}{3}y_1 + \frac{4}{3}y_2 + \cdots\right) - h_n\left(\frac{1}{3}y_0 + \qquad \frac{2}{3}y_2 + \cdots\right) \\
&= \frac{4}{3}h_n\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots\right) - \frac{1}{3}(2h_n)\left(\frac{1}{2}y_0 + y_2 + \cdots\right) \\
&= \frac{4}{3}T_n - \frac{1}{3}T_{\frac{n}{2}} = \frac{4T_n - T_{\frac{n}{2}}}{3}.
\end{aligned}$$

3. Improvements

$\mathcal{R}_{2,n} = B_n$, *Boole's Rule*, is obtained using the area of a quartic (degree 4 polynomial) approximation, is an exact approximation for quintics (degree 5 polynomials), requires n to be a multiple of 4, and uses the coefficients:

$$\frac{14}{45},\quad \frac{64}{45},\frac{24}{45},\frac{64}{45},\quad \frac{28}{45},\quad \frac{64}{45},\frac{24}{45},\frac{64}{45},\quad \frac{28}{45},\quad \ldots,\quad \frac{28}{45},\quad \frac{64}{45},\frac{24}{45},\frac{64}{45},\quad \frac{14}{45}.$$

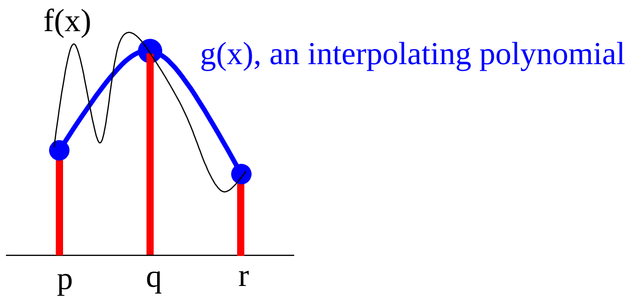
One can show

$$\mathcal{R}_{2,n} = \frac{16S_n - S_{\frac{n}{2}}}{15}.$$

The pattern can be used to define the *Richardson's extrapolates* (used in *Romberg Integration* by taking $m \to \infty$, with $n = 2^m$):

$$\mathcal{R}_{m,n} = \frac{4^m\mathcal{R}_{m-1,n} - \mathcal{R}_{m-1,\frac{n}{2}}}{4^m - 1}.$$

4. Derivation of Simpson's Rule via Interpolating Polynomials



The *Lagrange interpolating polynomial* which is a polynomial that passes through the same points as f at $x=p$, $x=q$ and $x=r$ is

$$g(x)=\frac{(x-q)(x-r)}{(p-q)(p-r)}f(p)+\frac{(x-p)(x-r)}{(q-p)(q-r)}f(q)+\frac{(x-p)(x-q)}{(r-p)(r-q)}f(r).$$

$\int_p^r f(x)\,dx$ is approximately

$$\int_p^r g(x)\,dx=\int_p^r \frac{(x-q)(x-r)}{(p-q)(p-r)}dx\cdot f(p)+\int_p^r \frac{(x-p)(x-r)}{(q-p)(q-r)}dx\cdot f(q)+\int_p^r \frac{(x-p)(x-q)}{(r-p)(r-q)}dx\cdot f(r).$$

In Simpson's rule we are interested in the case that $q-p=r-q=h$, that is, $q=p+h$ and $r=p+2h$. We show the last integral is $1/3$. Use substitution $u=\frac{x-p}{r-p}=\frac{x-p}{2h}$ so $2hu+p=x$. Then $2h\,du=dx$ and $\frac{x-q}{r-q}=\frac{2hu+p-q}{r-q}=\frac{2hu-h}{h}=2u-1$. The last integral becomes

$$\int_{u=0}^1 u(2u-1)2h\,du=h\int_{u=0}^1 4u^2-2u\,du=h\left(4\frac{u^3}{3}-u^2\right)\Big|_0^1=h\left(\frac{4}{3}-1\right)=h\frac{1}{3}.$$

By similar computations we could get the other two integrals, but there is an easier way. By symmetry the first one is also $h\frac{1}{3}$. The case $f(p)=f(q)=f(r)$ shows the three integrals must add to $2h$ so the middle one is $h\frac{4}{3}$.

5. Derivation of Simpson's Rule by attempting to cancel errors

Consider $f(x)=x^2$. On the one hand $\int_0^1 x^2\,dx=\frac{x^3}{3}\Big|_0^1=\frac{1}{3}$. On the other hand the Trapezoid approximation is

$$\begin{aligned}T_n&=\frac{1}{n}\left(\frac{1}{2}\frac{0^2}{n^2}+\frac{1^2}{n^2}+\cdots+\frac{1}{2}\frac{n^2}{n^2}\right)=\frac{1}{n}\left(\frac{1^2}{n^2}+\cdots+\frac{n^2}{n^2}-\frac{1}{2}\frac{n^2}{n^2}\right)=\frac{1}{n}\frac{1}{n^2}\left(1^2+\cdots+n^2-\frac{1}{2}n^2\right)\\&\overset{\text{sum}}{=}\frac{1}{n}\frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}-\frac{1}{2}n^2\right)\\&=\frac{1}{6n^2}\left((n+1)(2n+1)-3n^2\right)=\frac{1}{6n^2}\left(2n^2+1\right)=\frac{1}{3}+\frac{1}{6n^2}\end{aligned}$$

Similarly $T_{\frac{n}{2}}=\frac{1}{3}+\frac{1}{6(n/2)^2}=\frac{1}{3}+4\frac{1}{6n^2}$.

We want to average $T_n=\frac{1}{3}+E$ and $T_{\frac{n}{2}}=\frac{1}{3}+4E$ to cancel or reduce the term E which appears, where $E=\frac{1}{6n^2}$. Since $4T_n-T_{\frac{n}{2}}=1$ we see $\frac{4}{3}T_n-\frac{1}{3}T_{\frac{n}{2}}$ gives the exact value of $\frac{1}{3}$ for the integral. This expression, which is S_n, is a better approximation than the Trapezoid rule in this case.