A note on the Riemann sphere

Realise the Riemann sphere as the unit diameter sphere S^2 given in $\mathbf{R}^3$ as

$$u^2 + v^2 + (w - \tfrac{1}{2})^2 = \left(\tfrac{1}{2}\right)^2$$

and regard the uv-plane tangent to this sphere as $\mathbf{C}$, the complex z-plane.

Stereographic projection is the map from $\mathbf{C} \longrightarrow S^2$ determined by that point of intersection of the chord $\overline{zN}$ with S^2, where z is in the uv-plane and N is the north pole $(0,0,1)$ of S^2.

The line from N to $z = (u,v,0)$ has direction vector $(u,v,-1)$ and so has equation $\mathcal{L}(t) = N + t(u,v,-1)$ (here, we freely identify $z = u + iv$ with the ordered triple $(u,v,0)$ as required).

Observe that $\mathcal{L}(0) = N$ and $\mathcal{L}(1) = (u,v,0) = z$. The point of S^2 punctured by $\mathcal{L}$ will therefore have $0 < t < 1$. Denote this t by t_z. Since $\mathcal{L}(t_z)$ is a point of S^2 , this implies $\mathcal{L}(t_z) = (ut_z, vt_z, 1 - t_z)$ satisfies $(u^2 + v^2 + 1)t_z^2 - t_z = 0$. It follows that either $t_z = 0$ (giving the north pole of S^2) or $t_z = 1/(u^2 + v^2 + 1)$. This latter choice yields the intersection point

$$\mathcal{L}(t_z) = \frac{(u, v, u^2 + v^2)}{u^2 + v^2 + 1}$$

and defines the mapping of stereographic projection, sending the complex plane to the sphere.

This mapping from $\mathbf{C}$ to $S^2 \setminus N$ is given explicitly by

$$\rho : \mathbf{C} \longrightarrow S^2 \setminus N : z \longmapsto \frac{(\operatorname{Re} z, \operatorname{Im} z, |z|^2)}{1 + |z|^2}.$$

If d is the usual Euclidean metric in $\mathbf{C}$ and d' that on S^2 inherited from $\mathbf{R}^3$, then $d(z,w) = |z - w|$ and, after much arithmetic,

$$d(\rho(z), \rho(w)) = \frac{|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}} = \frac{d(z,w)}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}.$$

Evidently, $d'(\rho(z), \rho(w)) = 0$ iff $d(z,w) = 0$. Convergence properties in $\mathbf{C}$ and $S^2 \setminus N$ are therefore easily related. Observe that $z \to \infty$ implies $\rho(z) \to (0,0,1) = N$. Thus, convergence to ∞ in the complex plane corresponds to convergence to the north pole of the Riemann sphere.

The metric $d(\rho(z), \rho(w))$ is the *chordal metric* of the Riemann sphere S^2, regarded as the extended complex plane $\mathbf{C} \cup \{\infty\}$.