# ON COLOR PRESERVING AUTOMORPHISMS OF CAYLEY GRAPHS OF ODD SQUARE-FREE ORDER

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Abstract. An automorphism $\alpha$ of a Cayley graph $\mathrm{Cay}(G,S)$ of a group $G$ with connection set $S$ is color-preserving if $\alpha(g,gs)=(h,hs)$ or $(h,hs^{-1})$ for every edge $(g,gs)\in E(\mathrm{Cay}(G,S))$. If every color-preserving automorphism of $\mathrm{Cay}(G,S)$ is also affine, then $\mathrm{Cay}(G,S)$ is a CCA (Cayley color automorphism) graph. If every Cayley graph $\mathrm{Cay}(G,S)$ is a CCA graph, then $G$ is a CCA group. Hujdurović, Kutnar, D.W. Morris, and J. Morris have shown that every non-CCA group $G$ contains a section isomorphic to the nonabelian group $F_{21}$ of order 21. We first show that there is a unique non-CCA Cayley graph $\Gamma$ of $F_{21}$. We then show that if $\mathrm{Cay}(G,S)$ is a non-CCA graph of a group $G$ of odd square-free order, then $G=H\times F_{21}$ for some CCA group $H$, and $\mathrm{Cay}(G,S)=\mathrm{Cay}(H,T)\Box\Gamma$.
**Keywords.** Cayley graph, color-preserving automorphism, automorphism group, affine automorphism.

## 1. Introduction and preliminaries

We consider Cayley digraphs $\mathrm{Cay}(G,S)$ of a group $G$ with connection set $S$ whose arcs $(g,gs)$ are colored with the color $s$, for $s\in S$. It has been known since at least the early 1970s [12, Theorem 4-8] that any color-preserving automorphism of $\mathrm{Cay}(G,S)$ can only be an automorphism of $\mathrm{Cay}(G,S)$ induced by left translation by an element $k\in G$. That is, the only color-preserving automorphisms of $\mathrm{Cay}(G,S)$ are of the form $x\mapsto kx$ for some $k\in G$. The corresponding question for Cayley graphs $\mathrm{Cay}(G,S)$ was not considered until recently [8]. Note that the essential difference

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between the graph and digraph problem is that for a graph we insist that the connection set $S$ satisfies $S = S^{-1}$ and that we insist that any pair of arcs between two vertices $u$ and $v$ are colored with the *same* color (whereas in the digraph case we insist that the arcs between two vertices $u$ and $v$ are colored with *different* colors). It is easy to see that there can be additional color-preserving automorphisms of Cayley graphs. For example, if $G$ is an abelian group (written multiplicatively), then the map $x \mapsto x^{-1}$ is a color-preserving automorphism of any Cayley graph $\mathrm{Cay}(G, S)$. In this case, $x \mapsto x^{-1}$ is also a group automorphism of $G$, so it makes sense to ask if there are color-preserving automorphisms of $\mathrm{Cay}(G, S)$ that are not group automorphisms of $G$ or translations of $G$; that is, if there are color-preserving automorphisms of $\mathrm{Cay}(G, S)$ which are not **affine**. Hujdurović, Kutnar, D.W. Morris, and J. Morris showed [8] that non-affine color-preserving automorphisms of $\mathrm{Cay}(G, S)$ do exist for some groups $G$, and when $G$ is of odd order found fairly restrictive conditions for such a group $G$ to exist. In this paper we continue this program, and show that groups $G$ of odd square-free order $n$ for which there exists a Cayley graph $\mathrm{Cay}(G, S)$ with a color-preserving automorphism that is not affine have the form $H \times F_{21}$, where $H$ is group of order $n/21$ and $F_{21}$ is the nonabelian group of order 21. Additionally, we show that $\mathrm{Cay}(G, S)$ is the Cartesian product $\mathrm{Cay}(H, T) \,\square\, \Gamma$, where $T \subset H$ satisfies $T = T^{-1}$ and $\Gamma$ is the unique non-CCA Cayley graph of $F_{21}$.

Throughout this paper, all groups and graphs are finite.

**Definition 1.1.** Let $G$ be a group and $S \subset G$ such that $1_G \notin S$. Define a **Cayley digraph of** $G$, denoted $\mathrm{Cay}(G, S)$, to be the digraph with vertex set $V(\mathrm{Cay}(G, S)) = G$ and arc set $A(\mathrm{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}$. We call $S$ the **connection set of** $\mathrm{Cay}(G, S)$.

If $S = S^{-1}$ then $\mathrm{Cay}(G, S)$ is a graph, and we call the arc set the **edge set**. For $g \in G$, define $g_L : G \mapsto G$ by $g_L(x) = gx$. It is straightforward to verify that $G_L = \{g_L : g \in G\}$ is a group isomorphic to $G$. The group $G_L$ is the **left regular representation of** $G$ and it is an easy exercise to show that $G_L \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$.

**Definition 1.2.** Let $S \subset G$ such that $S = S^{-1}$ and to each pair $s, s^{-1} \in S$, assign a unique color $c(s) = c(s^{-1})$, so that $c(s') = c(s)$ implies $s' \in \{s, s^{-1}\}$. Let $S' = \{c(s) : s \in S\}$. Consider a Cayley graph $\mathrm{Cay}(G, S)$ in which each edge $(g, gs)$ is colored with $c(s) \in S'$, and $E_{c(s)}$ be the set of all edges of $\mathrm{Cay}(G, S)$ that are colored with the color $c(s) \in S'$. An automorphism $\alpha$ of $\mathrm{Cay}(G, S)$ is a **color-preserving** automorphism if $\alpha(E_{c(s)}) = E_{c(s)}$ for each $c(s) \in S'$. The set of all color-preserving automorphisms of $\mathrm{Cay}(G, S)$ is a group denoted $\mathcal{A}^o$.

Clearly $G_L \leq \mathcal{A}^o$, and if $\alpha \in \mathrm{Aut}(G)$, then $\alpha \in \mathcal{A}^o$ if and only if $\alpha(\{s, s^{-1}\}) = \{s, s^{-1}\}$ for every $s \in S$.

The group $G_L \cdot \mathrm{Aut}(G) \leq S_G$ (where $S_G$ denotes the symmetric group on $G$), is the normalizer in $S_G$ of $G_L$ by [3, Corollary 4.2B], and an element of $G_L \cdot \mathrm{Aut}(G)$ is called **affine**. As mentioned earlier, automorphism groups of Cayley graphs of abelian groups always contain an affine automorphism that is not in $G_L$, namely $x \mapsto x^{-1}$. Additionally, as an element $\alpha \in \mathrm{Aut}(G)$ is contained in

$\mathrm{Aut}(\mathrm{Cay}(G,S))$ and is color-preserving if and only if $\alpha(\{s,s^{-1}\}) = \{s,s^{-1}\}$ for every $s \in S$, given a Cayley graph $\mathrm{Cay}(G,S)$ is it straightforward to compute the subgroup of $G_L \cdot \mathrm{Aut}(G)$ that is color-preserving. Thus, the real challenge lies in determining whether or not a given Cayley graph has color-preserving automorphisms that are not affine.

**Definition 1.3.** We say a Cayley graph $\mathrm{Cay}(G,S)$ of a group $G$ is a CCA-graph if every color-preserving automorphism of $\mathrm{Cay}(G,S)$ is affine. A group $G$ is a CCA-group if and only if every connected Cayley graph of $G$ is a CCA-graph.

**Remark 1.4.** Since disconnected Cayley graphs will almost never be CCA-graphs, connectedness is important in this definition. We will refer to a graph as being non-CCA only if it is connected, so that it provides evidence that the corresponding group is not a CCA-group.

In [8, Theorem 6.8], the following constraints on groups of odd order that are not CCA were obtained. Before stating this result, we need another definition.

**Definition 1.5.** Let $G$ be a group. For any subgroups $H$ and $K$ of $G$, such that $K \triangleleft H$, the quotient $H/K$ is said to be a **section** of $G$.

**Theorem 1.6.** [8, Theorem 6.8] *Any non-CCA group of odd order has a section that is isomorphic to either:*

(1) *a semi-wreathed product $A \wr \mathbb{Z}_n$ (see* [8, Example 6.5] *for the appropriate definitions), where $A$ is a nontrivial, elementary abelian group (of odd order) and $n > 1$, or*
(2) *the (unique) nonabelian group of order* 21.

The groups in (1) do not have square-free order, so for the odd square-free integers $n$ under consideration in this paper, the preceding result says that they are CCA-groups unless they contain a section isomorphic to the nonabelian group $F_{21}$ of order 21. We now turn to improving this result, and begin with some preliminary results and the definitions needed for them.

**Definition 1.7.** Let $G$ be a transitive permutation group with invariant partition $\mathcal{B}$. By $G/\mathcal{B}$, we mean the subgroup of $S_{\mathcal{B}}$ induced by the action of $G$ on $\mathcal{B}$, and by $\mathrm{fix}_G(\mathcal{B})$ the kernel of this action. Thus $G/\mathcal{B} = \{g/\mathcal{B} : g \in G\}$ where $g/\mathcal{B}(B_1) = B_2$ if and only if $g(B_1) = B_2$, $B_1, B_2 \in \mathcal{B}$, and $\mathrm{fix}_G(\mathcal{B}) = \{g \in G : g(B) = B \text{ for all } B \in \mathcal{B}\}$. It is often the case that $G$ will have invariant partitions $\mathcal{B}$ and $\mathcal{C}$, and every block of $\mathcal{C}$ is a union of blocks of $\mathcal{B}$. In this case, we write $\mathcal{B} \preceq \mathcal{C}$. Then $G/\mathcal{B}$ admits an invariant partition with blocks consisting of those blocks of $\mathcal{B}$ contained in a block of $\mathcal{C}$. For $C \in \mathcal{C}$, we let $C/\mathcal{B}$ be the blocks of $\mathcal{B}$ that are contained in $C$, and write $\mathcal{C}/\mathcal{B}$ for $\{C/\mathcal{B} : C \in \mathcal{C}\}$.

**Proposition 1.8.** *Let $G$ be a group and $1_G \neq s \in S \subset G$. Then the left cosets of $\langle s \rangle$ in $G$ form an invariant partition of the color-preserving group of automorphisms $\mathcal{A}^o$ of $\mathrm{Cay}(G,S)$. Consequently, if there is some $s \in S$ such that $\langle s \rangle \neq G, \{1_G\}$, then $\mathcal{A}^o$ is imprimitive.*

*Proof.* The Cayley graph $\mathrm{Cay}(G,\{s^{\pm 1}\})$ has as its connected components the left cosets of $\langle s\rangle$ in $G$. If $a\in\mathcal{A}^o$, then $a(\mathrm{Cay}(G,\{s^{\pm 1}\}))=\mathrm{Cay}(G,\{s^{\pm 1}\})$ and $a$ certainly maps each connected component of $\mathrm{Cay}(G,\{s^{\pm 1}\})$ to some connected component of $\mathrm{Cay}(G,\{s^{\pm 1}\})$. So the left cosets of $\langle s\rangle$ form an invariant partition $\mathcal{B}$ of $\mathcal{A}^o$. Finally, the $\mathcal{A}^o$-invariant partition $\mathcal{B}$ is nontrivial, making $\mathcal{A}^o$ imprimitive, if and only if $s\neq 1_G$ does not generate $G$. □

**Definition 1.9.** Let $G\le S_X$ be a transitive permutation group, and $\mathcal{O}_0,\dots,\mathcal{O}_r$ the orbits of $G$ acting on $X\times X$. Assume that $\mathcal{O}_0$ is the diagonal orbit $\{(x,x):x\in X\}$. Define digraphs $\Gamma_1,\dots,\Gamma_r$ by $V(\Gamma_i)=X$ and $A(\Gamma_i)=\mathcal{O}_i$. The graphs $\Gamma_1,\dots,\Gamma_r$ are the orbital digraphs of $G$. Define the **2-closure of** $G$, denoted by $G^{(2)}$, as $\cap_{i=1}^r\mathrm{Aut}(\Gamma_i)$. We say $G$ is **2-closed** if $G^{(2)}=G$.

Observe that if the arcs of each $\Gamma_i$ are colored with color $i$, $1\le i\le r$, then $G^{(2)}$ is the automorphism group of the resulting color digraph. Therefore, we can equivalently define that $G$ is 2-closed if and only if $G$ is the automorphism group of a colour (di)graph. This is the definition we will use in our next proof.

**Lemma 1.10.** *For a Cayley graph* $\mathrm{Cay}(G,S)$*, the group* $\mathcal{A}^o$ *is* 2*-closed.*

*Proof.* We show that $\mathcal{A}^o$ is the automorphism group of a Cayley color graph. This is done in the obvious way. Namely, we edge-color $\mathrm{Cay}(G,S)$ with the natural edge-coloring described in Definition 1.2. It is then clear that $\mathcal{A}^o$ is the automorphism group of the resulting Cayley color graph. □

## 2. CCA graphs of $F_{21}$ and complete CCA graphs

In this section, we will show that $\mathrm{Cay}(F_{21},\{a^{\pm 1},(ax)^{\pm 1}\})$ is the unique non-CCA graph of $F_{21}$, where $F_{21}=\langle a,x|a^3=x^7=e,a^{-1}xa=x^2\rangle$ is the nonabelian group of order 21. This example was first given in [8, Example 2.3], and is drawn in Fig. 1 (note that $(ax)^{\pm 1}=\{x^4a,x^6a^2\}$). Edges corresponding to colors $a^{\pm 1}$ are in black, while edges corresponding to $(ax)^{\pm 1}$ are in red. We will make use of [5, Theorem 3.2], and observe that while this result is stated for graphs, the proofs hold for digraphs $\Gamma$ and 2-closed groups $G$ provided that $\mathrm{Aut}(\Gamma)$ (or $G$) has a nontrivial invariant partition formed by the orbits of a normal subgroup (but the result does not hold for digraphs or 2-closed groups if this condition is not satisfied).

In order to show that $\mathrm{Cay}(F_{21},\{a^{\pm 1},(ax)^{\pm 1}\})$ is the unique non-CCA graph of $F_{21}$, we require a separate argument to show that the complete Cayley graph on $F_{21}$ is CCA. Since this argument generalizes fairly straightforwardly to any complete Cayley graph on a group that is not a Hamiltonian 2-group, we present the generalization here. We begin with a couple of results we will need repeatedly in the proof.

**Lemma 2.1.** *Let* $\Gamma=\mathrm{Cay}(G,G\setminus\{1_G\})$ *be a complete graph, viewed as a Cayley graph on a group* $G$*, and let* $\varphi$ *be a color-preserving automorphism that fixes* $1_G$*. If* $g,x\in G$ *with* $\varphi(x)=x^{-1}\neq x$

*and* $\varphi(g) = g$*, then* $x^{-1}gx = g^{-1}$*. Furthermore, if* $\varphi$ *does not invert every element of* $G$*, then* $|x| = 4$*.*

*Proof.* Let $h = g^{-1}x$, so that $x = gh$. Now, $x \sim g$ via an edge of color $c(h)$ (where $x \sim g$ means there is an edge between $x$ and $g$), so $\varphi(x)$ must be adjacent to $\varphi(g) = g$ via an edge of color $c(h)$, meaning $\varphi(x) \in \{gh, gh^{-1}\}$. Thus, $(gh)^{-1} \in \{gh, gh^{-1}\}$. If $|h| = 2$ so that $gh^{-1} = gh$ or $(gh)^{-1} = gh$, then $x^{-1} = \varphi(x) = x$, contradicting our choice of $x$. So we must have $|h| > 2$ and $(gh)^{-1} = gh^{-1}$, i.e., $gh^{-1}gh = 1_G$, so $h^{-1}gh = g^{-1}$. Hence $x^{-1}gx = h^{-1}gh = g^{-1}$.

Observe that for every $a \in G$, either $a = 1_G$ or $a \sim 1_G$. Since $\varphi$ is color-preserving, this means $\varphi(a) \in \{a, a^{-1}\}$. So if $\varphi$ does not invert every element of $G$ then we may assume without loss of generality that it fixes some $g$ with $|g| > 2$. Let $y = gh^2$. Since $y \neq g$ and $y \sim x$ via an edge of color $c(h)$, we must have $\varphi(y) = \varphi(x)h$ or $\varphi(y) = \varphi(x)h^{-1}$. This implies that $\varphi(y) = g$ or $\varphi(y) = gh^{-2}$. Since by the assumption, $\varphi(g) = g$ and $\varphi$ is bijection, it follows that $\varphi(g) = gh^{-2}$. If $\varphi(y) = y^{-1}$, then $gh^{-2}gh^2 = 1_G$, but this contradicts $h^{-1}gh = g^{-1}$ since $|g| > 2$. So we must have $|h| = 4$ and $\varphi(y) = y$. Now since $h^{-1}gh = g^{-1}$ and $x = gh$, we see that $x^2 = h^2$, so $|x| = 4$. Also, $x^{-1}gx = h^{-1}gh = g^{-1}$. Thus, for any $g, x \in G$ with $\varphi(g) = g$ and $\varphi(x) = x^{-1}$, we have $x$ inverts $g$ and $|x| = 4$. □

As was mentioned above, Hamiltonian 2-groups play an important role in this statement. We give here their definition and some key facts.

**Definition 2.2.** A **Hamiltonian 2-group** is a nonabelian 2-group, all of whose subgroups are normal. It was proven by Dedekind in the finite case, and extended by Baer to the infinite case (and is now well-known), that Hamiltonian 2-groups have the form $Q_8 \times \mathbb{Z}_2^n$ for some non-negative integer $n$.

**Theorem 2.3.** *Let* $\Gamma = \mathrm{Cay}(G, G \setminus \{1_G\})$ *be a complete graph, viewed as a Cayley graph on a group* $G$*. Then* $\Gamma$ *is a CCA graph if and only if* $G$ *is not a Hamiltonian* 2*-group.*

*Proof.* First suppose that $G$ is a Hamiltonian 2-group. Define $\varphi$ by $\varphi(x) = x^{-1}$ for every $x \in G$. Since Hamiltonian 2-groups are nonabelian, $\varphi$ is not a group automorphism of $G$. To show that $\varphi$ is color-preserving, let $x, y \in G$ with $y = xh$. Then $\varphi(y) = y^{-1} = h^{-1}x^{-1}$. In $G \cong Q_8 \times \mathbb{Z}_2^n$, every element either inverts or commutes with every other element, so $h^{-1}x^{-1} = x^{-1}h^{\pm 1} = \varphi(x)h^{\pm 1}$. Hence the edge from $\varphi(x)$ to $\varphi(y)$ has the same color, $c(h)$, as the edge from $x$ to $y$. This shows that $\Gamma$ is not a CCA graph.

For the converse, let $\varphi$ be an arbitrary color-preserving automorphism of $\Gamma$ that fixes the vertex $1_G$. We will show that either $G$ is a Hamiltonian 2-group, or $\varphi$ is a group automorphism of $G$.

Suppose initially that for every $g \in G$, $\varphi(g) = g^{-1}$. If $G$ is abelian then $\varphi$ is an automorphism of $G$, so we suppose that there exist $g, h$ such that $gh \neq hg$. The fact that $\varphi$ is color-preserving forces $(gh)^{-1} = \varphi(gh) \in \{\varphi(g)h, \varphi(g)h^{-1}\} = \{g^{-1}h, g^{-1}h^{-1}\}$. Since $g$ and $h$ do not commute, we see that $(gh)^{-1} \neq g^{-1}h^{-1}$ (so $(gh)^{-1} = g^{-1}h$) and that $|h| > 2$. Similarly, reversing the roles of

$g$ and $h$, we conclude $(hg)^{-1} = h^{-1}g$ and $|g| > 2$. Thus, $h$ and $g$ invert each other. Furthermore, combining these yields $gh = g^{-1}h^{-1}$, so $g^2 = h^{-2}$. But we also have $(gh)^2 = ghgh = g^2 = h^2$, so $h^2 = h^{-2}$, meaning $|h| = |g| = 4$. Observe that since every pair of non-commuting elements invert each other, every subgroup of $G$ is normal, so that (since $G$ is nonabelian) $G$ is Hamiltonian. We have also shown that every element not in the centre of $G$ has order 4; if $z$ is in the centre of $G$, then $gz$ does not commute with $h$, so $gz$ has order 4. Furthermore, since every pair of non-commuting elements have equal squares, $(gz)^2 = g^2z^2 = h^2$ and since $g^2 = h^2$, we see that $|z| = 2$. Thus, $G$ is a Hamiltonian 2-group.

We may now assume that there is some $g \in G \setminus \{1_G\}$ such that $\varphi(g) = g$ and $|g| > 2$. If $\varphi = 1$ then we are done, so there must be some $x \in G$ such that $\varphi(x) = x^{-1} \neq x$.

Suppose that the elements fixed by $\varphi$ form a nontrivial proper subgroup $H$ of $G$; that is, whenever $g_1, g_2 \in G$ are fixed by $\varphi$, then so is $g_1g_2$. Since $\varphi(x) = x^{-1}$, by Lemma 2.1, $x$ inverts $g_1, g_2$, and $g_1g_2$, so $x^{-1}g_1g_2x = x^{-1}g_1xx^{-1}g_2x = g_1^{-1}g_2^{-1} = (g_1g_2)^{-1}$. This implies that $g_1$ and $g_2$ commute. Thus, $H$ is abelian. Since $|x| = 4$ (by Lemma 2.1), $x^2$ is fixed by $\varphi$, so $x^2 \in H$. If $H$ has index 2 in $G$ then by the definition of $H$, every $xh \in xH$ is inverted by $\varphi$. It is straightforward to show that $\varphi$ is a group automorphism of $G$ in this case (in fact, such a $G$ is a generalised dicyclic group, and this map $\varphi$ is a well-known automorphism of such groups). If the index of $H$ is greater than 2 then there is some $y \notin H \cup xH$, so $y$ is inverted by $\varphi$. Furthermore, since $y \notin xH = x^{-1}H$, we see that $xy \notin H$, so $xy$ is also inverted by $\varphi$. But then $x, y$, and $xy$ all invert every $g \in H$, which is not possible (if $x$ and $y$ invert $g$ then $xy$ commutes with $g$).

We may now assume that the elements fixed by $\varphi$ do not form a subgroup of $G$, so there exist $a, b \in G$ with $\varphi(a) = a$, $\varphi(b) = b$, and $\varphi(ab) = (ab)^{-1}$. By Lemma 2.1, $ab$ inverts $a$ and $b$, so $a$ and $b$ invert each other, and $|ab| = 4$. Hence $a^2 = b^2 = (ab)^2$. This is enough to characterise $\langle a, b\rangle \cong Q_8$. Relabel with the standard notation for $Q_8$, so $i = a$ and $j = b$. We will show that for every element $y$ of $G$, $y^2 = \pm 1$.

First, let $h$ be an arbitrary element of $G$ that is not in $\langle i, j\rangle$, and that is inverted by $\varphi$. By Lemma 2.1, $|h| = 4$ and $h$ inverts $i$ and $j$. Also, since $h$ inverts $i$ and $j$ and $k = ij$, we see that $h$ must commute with $k$. Observe that if $hk$ were inverted by $\varphi$, then $hk$ would invert $i$ (and $j$), but this is impossible since $h$ and $k$ each invert $i$. So $hk$ must be fixed by $\varphi$. Thus $h$ and $k$ both invert $hk$, which implies that $h^2 = k^2 = -1$, as desired.

Let $g$ be an arbitrary element of $G$ that is not in $\langle i, j\rangle$, and that is fixed by $\varphi$. If $|g| = 2$ then $g^2 = 1$, as claimed. So we assume $|g| > 2$. Suppose that $gi$ is fixed by $\varphi$. Then $k$ inverts $gi$, so $k^{-1}gik = i^{-1}g^{-1} = g^{-1}i^{-1}$ since $k$ inverts both $g$ and $i$, so $i$ and $g$ commute. The same is true for $gj$; thus, if $gi$ and $gj$ were both fixed by $\varphi$, then $g$ would commute with $i$ and $j$ and hence with $k$, a contradiction since $|g| > 2$. So at least one of $gi$ and $gj$ is inverted by $\varphi$. Suppose that $gi$ is inverted by $\varphi$. By the argument of the previous paragraph, $(gi)^2 = -1$. Also, $gi$ inverts $i$, so $g$ inverts $i$. Hence $-1 = (gi)^2 = g(gi^{-1})i = g^2$, completing the proof of our claim.

We have shown that every non-identity element of $G$ has order 2 or 4 (so $G$ is a 2-group), and that the elements of order 4 all square to $-1$. To complete the proof that $G$ is Hamiltonian, we need to show that every subgroup is normal. Let $r, s \in G$. Then $(rs)^2 = \pm 1$, so $rsr = \pm s$, so $r^{-1}sr = rsr$ if $r$ has order 2 or $r^{-1}sr = -rsr$ otherwise. In any case, $r^{-1}sr = \pm s$. If $r^{-1}sr = s$ then $\langle s \rangle \trianglelefteq G$. If $s$ is of order 4, then $-s = -1 \cdot s = s^{-1}$, which implies $r^{-1}sr \in \langle s \rangle$ and therefore, $\langle s \rangle \trianglelefteq G$. Suppose now that $s$ is of order two and $r^{-1}sr = -s$. Observe that $\varphi(s) = s$. By Lemma 2.1, we must have $\varphi(r) = r$. If $\varphi(rs) = (rs)^{-1}$, then by Lemma 2.1, it follows that $(rs)^{-1}s(rs) = s^{-1} = s$, so that $r^{-1}sr = s$, a contradiction. If $\varphi(rs) = rs$, then using the fact that $\varphi(k) = k^{-1} \neq k$, and Lemma 2.1, it follows that $k$ inverts $r$, $s$ and $rs$, but this implies that $r$ and $s$ commute, a contradiction. So we in fact have $r^{-1}sr = s^{\pm 1}$, and $\langle s \rangle \trianglelefteq G$. Similarly, $\langle r \rangle \trianglelefteq G$. Thus every subgroup is normal in $G$, so $G$ is Hamiltonian. □

For the reader's convenience, we state the parts we will be using of what is proved in [5, Theorem 3.2]. We will need some additional notation and terminology.

**Definition 2.4.** Let $q < p$ be prime, $\alpha \in \mathbb{Z}_p^*$, and define $\rho, \tau : \mathbb{Z}_q \times \mathbb{Z}_p \mapsto \mathbb{Z}_q \times \mathbb{Z}_p$ be given by $\rho(i,j) = (i, j+1)$ and $\tau(i,j) = (i+1, \alpha j)$. Let $G \leq S_{pq}$ contain $\langle \rho, \tau \rangle$ but not a regular cyclic subgroup. We will say that $\alpha \in \mathbb{Z}_p^*$ is *maximal* in $G$ if $\tau(i,j) = (i+1, \alpha j)$ and whenever $\tau'(i,j) = (i+1, \beta j)$ for some $\beta \in \mathbb{Z}_p^*$ and $\tau' \in G$, then $\langle \rho, \tau' \rangle \leq \langle \rho, \tau \rangle$.

We remark that the existence of a maximal $\alpha$ follows from [5, Theorem 2.7].

**Theorem 2.5** ([5, Theorem 3.2])**.** *Let $G$ be a 2-closed group of degree $pq$, $p > q$, admitting a nontrivial invariant partition $\mathcal{B}$, and suppose that $G$ does not contain a regular cyclic subgroup. Then*

(1) *if $G$ is the automorphism group of a graph, and every nontrivial invariant partition $\mathcal{B}$ admitted by $G$ consists of $p$ blocks of size $q$ then $p = 2^{2^s} + 1$ is a Fermat prime (and $q$ divides $p - 2$).*

(2) *if $\mathcal{B}$ is formed by the orbits of a normal subgroup then $\mathcal{B}$ consists of $q$ blocks of size $p$, and $M = \langle \rho, \tau \rangle \leq G$ for $\alpha$ maximal in $G$. If $\alpha$ has order $a$, then*
   - (a) *if $\mathrm{fix}_G(\mathcal{B})|_B$ is doubly transitive for $B \in \mathcal{B}$, then $q = 2$ and*
     - (i) *$G = \mathbb{Z}_2 \ltimes \mathrm{PSL}(2, 11)$, or*
     - (ii) *$G = \mathbb{Z}_2 \ltimes \mathrm{P\Gamma L}(n, k)$, where $n$ is prime, $k = r^m$, $r$ is prime, $\gcd(n, k-1) = 1$, and $m$ is a power of $n$; while*
   - (b) *if $\mathrm{fix}_G(\mathcal{B})|_B$ is not doubly transitive, then*
     - (i) *if $a \neq q$ then $G = M$, and*
     - (ii) *if $a = q$ then $M \lhd G$.*

It should be noted that in the proof of part (1) of [5, Theorem 3.2], it is deduced that $\mathrm{fix}_G(\mathcal{B}) = 1$, so that $\mathcal{B}$ is not formed by the orbits of a normal subgroup, so the statement above reflects the fact that only part (2) can arise if $\mathcal{B}$ is formed by the orbits of a normal subgroup. Furthermore,

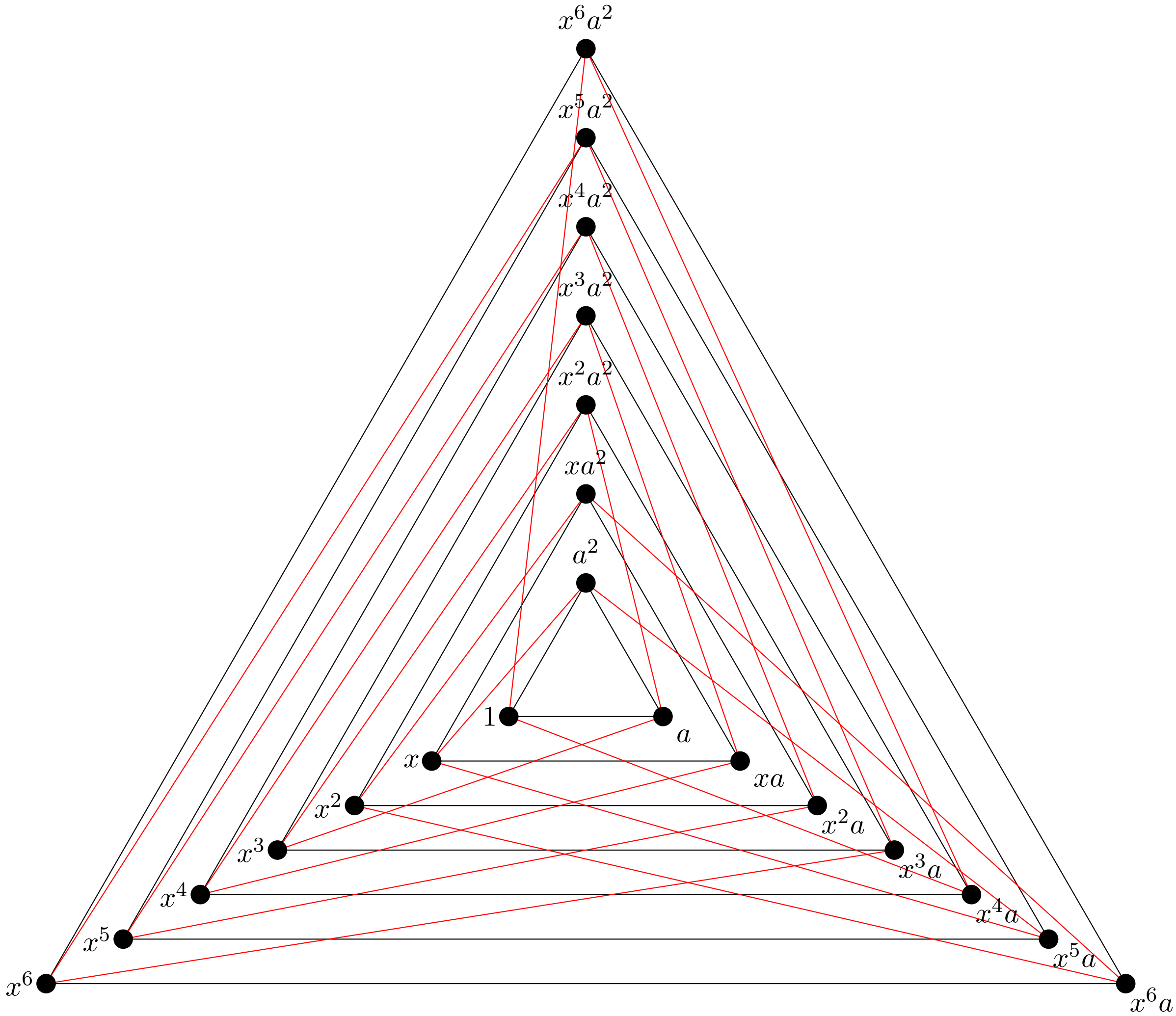


Figure 1. The unique non-CCA Cayley graph of $F_{21}$.

what we have stated for part (2bii) requires combining the statements of Lemma 2.5 and Theorem 3.2 (2bii) of [5]. Additionally, the proofs given in [5, Theorem 3.2] hold for digraphs as well as for 2-closed groups as a 2-closed group can be written as the intersection of automorphism groups of digraphs. We caution the reader though that results concerning graphs with quasiprimitive automorphism group that are cited in the proof of [5, Theorem 3.2] need not be true for digraphs or 2-closed groups. For example, in the next result we will encounter the group $\mathrm{PSL}(2,7)$ which has an imprimitive representation of degree 21 with 7 blocks of size 3 and is the automorphism group of a digraph [11, Lemma 4.8], and so is 2-closed. However, 7 is not a Fermat prime.

**Proposition 2.6.** *The Cayley graph* $\mathrm{Cay}(F_{21}, \{a^{\pm 1}, (ax)^{\pm 1}\})$ *where* $a^3 = x^7 = e$ *and* $a^{-1}xa = x^2$ *is the only non-CCA graph of* $F_{21}$ *up to isomorphism. It is regular of valency* 4*, is an orbital* *graph of the primitive group* $\mathrm{PGL}(2,7)$*, and has imprimitive color-preserving automorphism group* $\mathcal{A}^o = \mathrm{PSL}(2,7)$.

*Proof.* Let $\Gamma = \mathrm{Cay}(F_{21}, S)$ be a non-CCA graph of $F_{21}$. As noted in Remark 1.4, we are only considering connected graphs. Then $(F_{21})_L \leq \mathcal{A}^o$ is not normal in $\mathcal{A}^o$, which is 2-closed by Lemma 1.10. As 7 is not a Fermat prime, by Theorem 2.5(1) $\mathrm{Aut}(\Gamma)$ is either primitive, contains a regular cyclic subgroup, or has an invariant partition $\mathcal{B}$ with blocks of size 7 formed by the orbits of a normal subgroup. Observe that in the latter case if $G$ is any transitive subgroup of $\mathrm{Aut}(\Gamma)$, then $G$ contains at least one element of order 7. Since $G/\mathcal{B} \leq S_3$ has order coprime to 7, every element of order 7 must lie in $\mathrm{fix}_G(\mathcal{B})$, so (using transitivity) the orbits of $\mathrm{fix}_G(\mathcal{B})$ are the blocks of $\mathcal{B}$.

Suppose that $\mathcal{A}^o$ does not contain a regular cyclic subgroup and $\mathrm{Aut}(\Gamma)$ is not primitive. Then $\mathcal{A}^o$ has an invariant partition $\mathcal{B}$ with blocks of size 7 formed by the orbits of a normal subgroup. As 21 is odd, it cannot be the case that Theorem 2.5(2a) occurs. If Theorem 2.5(2bii) occurs then $(F_{21})_L \triangleleft \mathcal{A}^o$, a contradiction. Otherwise, Theorem 2.5(2bi) implies that $\mathcal{A}^o$ contains a normal Sylow $p$-subgroup $P$ and $\mathcal{A}^o/P$ is cyclic and so contains a unique normal subgroup of order $q$. We conclude $(F_{21})_L \triangleleft \mathcal{A}^o$, a contradiction. Thus either $\mathcal{A}^o$ contains a regular cyclic subgroup or $\mathrm{Aut}(\Gamma)$ is primitive.

Let $R \leq \mathcal{A}^o$ be any regular subgroup. Since $\mathrm{Stab}_{\mathcal{A}^o}(x)$ is a 2-group for every $x \in V(\Gamma)$, it follows that $|\mathcal{A}^o| = 3 \cdot 7 \cdot 2^\ell$ for some integer $\ell$. So a Sylow 7-subgroup of $\mathcal{A}^o$ has order 7, and a Sylow 3-subgroup of $\mathcal{A}^o$ has order 3. After appropriate conjugations of $R$, if necessary, we may assume that $\langle R, (F_{21})_L\rangle$ has a unique Sylow 7-subgroup, $\langle x\rangle$, and that $R$ contains the Sylow 3-subgroup $\langle a\rangle$. Since $|R| = 21$ we must have $R \leq \langle a, x\rangle = (F_{21})_L$. In particular, $R$ cannot be cyclic.

Finally, suppose that $\mathrm{Aut}(\Gamma)$ is primitive. We know that $\mathrm{Aut}(\Gamma)$ is not doubly-transitive because this would imply $\Gamma$ being complete (since it is connected, it cannot be empty), and Theorem 2.3 shows that the complete Cayley graph on $F_{21}$ is CCA. There are only two primitive permutation groups of degree 21: $\mathrm{PGL}(2,7)$, and $A_7$ or $S_7$ in their action on unordered pairs [3, Appendix B]. But $S_7$ and $A_7$ in this action have stabilizer of order a multiple of $7!/42$ which is not a power of a 2, contradicting [8, Lemma 6.3]. So we must have $\mathrm{Aut}(\Gamma) \cong \mathrm{PGL}(2,7)$. We know that $\mathcal{A}^o$ contains $F_{21} \leq \mathrm{PSL}(2,7) \triangleleft \mathrm{PGL}(2,7)$. Since $F_{21}$ is a maximal subgroup of $\mathrm{PSL}(2,7)$ by the ATLAS of Finite Group Representations [10], we have $\mathcal{A}^o \cap \mathrm{PSL}(2,7)$ is either $F_{21}$ or $\mathrm{PSL}(2,7)$.

Suppose $\mathcal{A}^o \cap \mathrm{PSL}(2,7) = (F_{21})_L$. Then $\mathcal{A}^o$ is one of the maximal subgroups of $\mathrm{PGL}(2,7)$ that do not contain $\mathrm{PSL}(2,7)$; but these are $F_{21}$ and $F_{42} = \mathbb{Z}_7 \rtimes \mathbb{Z}_6$, and $F_{21}$ is normal in each of these, so such a graph would be CCA. So we must have $\mathcal{A}^o \cap \mathrm{PSL}(2,7) = \mathrm{PSL}(2,7)$. Since $\mathrm{PSL}(2,7)$ is maximal (of index 2 in fact) in $\mathrm{PGL}(2,7)$, and $\mathrm{PGL}(2,7)$ is primitive but $\mathcal{A}^o$ is imprimitive by Proposition 1.8, we must have $\mathcal{A}^o \cong \mathrm{PSL}(2,7)$.

Now, by [8, Example 2.3] $\Gamma_1 = \mathrm{Cay}(F_{21}, \{a^{\pm 1}, (ax)^{\pm 1}\})$ is a non-CCA graph of $F_{21}$ and of course has valency 4. By the above argument, we must have $\mathrm{Aut}(\Gamma_1) = \mathrm{PGL}(2,7)$. As $\mathrm{PGL}(2,7)$ has three suborbits of lengths $4, 8, 8$ by the ATLAS of Finite Group Representations [10], we conclude that $\Gamma_1$ is an orbital graph of $\mathrm{PGL}(2,7)$. One can then check, for example with MAGMA [2], that $\Gamma_1$ is the only Cayley graph of $F_{21}$ with automorphism group $\mathrm{PGL}(2,7)$ that is not a CCA graph of $F_{21}$. $\square$

**Remark 2.7.** While the Cayley graph $\Gamma = \mathrm{Cay}(\mathbb{F}_{21}, \{a^{\pm 1}, (ax)^{\pm 1}\})$ given in preceding Proposition is unique up to isomorphism, there are 21 different choices for $S$ which will yield a graph isomorphic to $\Gamma$, each of which is the image of $\Gamma$ under an automorphism of $\mathbb{F}_{21}$ by an element of the unique subgroup of $\mathrm{Aut}(\mathbb{F}_{21})$ of order 21. To see this, recall that $\mathrm{Aut}(\Gamma) \cong \mathrm{PGL}(2,7)$ is primitive. Now, $\mathbb{Z}_{21}$ is a Burnside group, so that a primitive group containing a regular copy of $\mathbb{Z}_{21}$ must be doubly-transitive, which $\mathrm{Aut}(\Gamma)$ clearly is not. Thus, $\mathrm{Aut}(\Gamma)$ does not contain a regular cyclic subgroup. Furthermore, $7^2$ does not divide $|\mathrm{Aut}(\Gamma)| = 336$. Thus, by [4, Theorem 9] any Cayley graph of $\mathbb{F}_{21}$ is isomorphic to $\Gamma$ if and only if an isomorphism between the two graphs is in $\mathrm{Aut}(\mathbb{F}_{21})$. It is not hard to show this group has order 42 (it is isomorphic to $F_{42} = \mathbb{Z}_7 \rtimes \mathbb{Z}_6$, and consists of the inner automorphisms together with the outer automorphism of order 2), and that $\mathrm{Aut}(\mathbb{F}_{21}) \cap \mathrm{Aut}(\Gamma)$ has order 2 (only the outer group automorphism acts as a graph automorphism).

## 3. Structure of non-CCA graphs of odd square-free order

**Definition 3.1.** The **Cartesian product** $G \,\square\, H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and edge set $\{(u,v)(u',v') : u = u' \text{ and } u'v' \in E(H) \text{ or } v = v' \text{ and } uu' \in E(G)\}$.

**Definition 3.2.** Let $G$ be a group and $K \leq G$. We denote the subgroup $\{k_L : k \in K\}$ of $G_L$ by $\hat{K}_L$.

We now give a sufficient condition for a Cayley color graph (with the natural edge-coloring defined in Definition 1.2) of odd order to be a Cartesian product. This will be a crucial tool in our main result.

**Lemma 3.3.** *For a connected Cayley color graph (with the natural edge-coloring defined in Definition 1.2)* $\mathrm{Cay}(G,S)$ *of a group* $G$ *of odd order, let* $\mathcal{B}$ *be an* $\mathcal{A}$*-invariant partition where* $G_L \leq \mathcal{A} \leq \mathcal{A}^o$ *is transitive and* $\mathcal{B}$ *is formed by the orbits of* $(\hat{G_2})_L \triangleleft G_L$*. Define an equivalence relation* $\equiv$ *on* $G$ *by* $g \equiv h$ *if and only if* $\mathrm{Stab}_{\mathrm{fix}_{\mathcal{A}}(\mathcal{B})}(g) = \mathrm{Stab}_{\mathrm{fix}_{\mathcal{A}}(\mathcal{B})}(h)$*. Let* $\Gamma$ *be the graph obtained from* $\mathrm{Cay}(G,S)$ *by removing all edges both of whose endpoints are contained in some block* $B \in \mathcal{B}$*. The following hold:*

1. *the equivalence classes of* $\equiv$ *form an* $\mathcal{A}$*-invariant partition* $\mathcal{E}$*,*
2. *each connected component of* $\Gamma$ *is contained in some* $E \in \mathcal{E}$*,*
3. *if each* $E \in \mathcal{E}$ *contains exactly one element from each* $B \in \mathcal{B}$*, then let* $S_2 = S \cap G_2$*,* $S_1 = S \setminus S_2$*, and* $G_1 = \langle S_1 \rangle$*. If* $G = G_1 \times G_2$*, then* $\mathrm{Cay}(G,S) \cong \Gamma_1 \,\square\, \Gamma_2$*, where* $\Gamma_1$ *is a connected component of* $\Gamma$ *and* $\Gamma_2 = \mathrm{Cay}(G,S)[B_1]$*, with* $B_1 \in \mathcal{B}$*. Furthermore,* $\Gamma_1 \cong \mathrm{Cay}(G_1,S_1)$*, and* $\Gamma_2 = \mathrm{Cay}(G_2,S_2)$*, and the natural edge-coloring given in Definition 1.2 is preserved under the isomorphism.*

*Proof.* As $G_L \leq \mathcal{A}$, any invariant partition of $\mathcal{A}$ is also a $G_L$-invariant partition. The partition $\mathcal{E}$ is an $\mathcal{A}$-invariant partition as the equivalence relation $\equiv$ is an $\mathcal{A}$-congruence [3, Exercise 1.5.4]. Now let $B_g \in \mathcal{B}$ with $g \in B_g$, and $h = gs \in G$ such that $h \notin B_g$ and $s \in S$ (so $(g, gs) \in E(\Gamma)$). Then $h \in B_h \in \mathcal{B}$ for some $B_h \neq B_g$.

Let $\alpha \in \mathrm{Stab}_{\mathrm{fix}_{\mathcal{A}}(\mathcal{B})}(g)$. We first claim that $\alpha(h) = h$. Indeed, as $\alpha \in \mathcal{A}^o$ and $\alpha(g) = g$, we have that $\alpha(h) = h = gs$ or $\alpha(h) = gs^{-1}$. If $\alpha(h) = gs^{-1}$, then as $\alpha(B) = B$ for all $B \in \mathcal{B}$, it must be the case that $gs^{-1}, gs \in B_h$. So there exists $k \in G_2$ such that $gsk = gs^{-1}$ so that $s^2 = k^{-1}$. As $k^{-1} \in G_2$ while $s \notin G_2$ we have that $\langle k \rangle < \langle s \rangle$ and as $s^2 \in \langle k \rangle$ we see that the order of $s$ is even. However, $G$ has odd order, a contradiction. Thus $\alpha(h) = h$ completing the claim.

The claim implies that if $\alpha \in \mathrm{fix}_{\mathcal{A}}(\mathcal{B})$ fixes $1_G$, then it fixes the neighbors of $1_G$ not contained in $B \in \mathcal{B}$ with $1_G \in B$. Arguing inductively, $\alpha$ fixes all vertices that are words formed by elements contained in $S_1 = S \backslash B$. We conclude that $\alpha$ fixes every vertex in the connected component of $\Gamma$ that contains $1_G$. Clearly all of these vertices are contained in the equivalence class of $\equiv$ that contains $1_G$, and (2) follows. This also shows that the vertices of this connected component consist of the subgroup $G_1$ of $G$ that is generated by $S_1$.

Now we additionally assume that $\mathrm{Stab}_{\mathrm{fix}_{\mathcal{A}}(\mathcal{B})}(1_G)$ fixes exactly one element from every block of $\mathcal{B}$. We have seen that $\Gamma_1 = \mathrm{Cay}(G_1, S_1)$ where $S_1 = S \setminus B$, and $\Gamma_2 = \mathrm{Cay}(G_2, S_2)$ where $S_2 = S \cap B$, so $S = S_1 \cup S_2$. For each $g \in G$, let $B_g \in \mathcal{B}$ contain $g$ and $E_g \in \mathcal{E}$ contain $g$. As $\mathrm{Cay}(G, S)$ is connected and $G$ is transitive, it must be the case that $\Gamma_1$ is isomorphic to the induced subgraph on $E_g$, and that $\Gamma_2$ is connected. Also, an element $g \in G$ is uniquely determined by the pair $(E_g, B_g) \in \mathcal{E} \times \mathcal{B}$ where $g \in E_g$ and $g \in B_g$. Define $\delta : G \to \mathcal{E} \times \mathcal{B}$ by $\delta(g) = (E_g, B_g)$. As $V(\Gamma_1) = E_i$ for some $i$ and $|E_i \cap B| = 1$ for every $B \in \mathcal{B}$, we identify each vertex of $V(\Gamma_1)$ uniquely with the block of $\mathcal{B}$ it is contained in. Similarly, as $V(\Gamma_2) = B_1$, we identify each vertex of $V(\Gamma_2)$ with the block of $\mathcal{E}$ that it is contained in. We claim that $\delta$ is an isomorphism between $\mathrm{Cay}(G, S)$ and $\Gamma_1 \,\Box\, \Gamma_2$ that preserves the edge colors.

Let $e = gh \in E(\mathrm{Cay}(G, S))$. As every edge of $\mathrm{Cay}(G, S)$ is either contained in $\Gamma$ (recall that $\Gamma$ is the graph obtained from $\mathrm{Cay}(G, S)$ by removing all edges both of whose endpoints are contained in some block $B \in \mathcal{B}$) or $\mathrm{Cay}(G, S) \backslash E(\Gamma)$, both endpoints of $e$ are contained in a component of $\Gamma$ (which is the same as a block of $\mathcal{E}$) or a block of $\mathcal{B}$. In the former case, $\delta(gh) = (E_g, B_g)(E_h, B_h)$ and $E_g = E_h$ while $B_g B_h \in E(\Gamma_1)$, so that $h = gs$ for some $s \in S_1$ and this color is preserved by the isomorphism. In the latter case, $\delta(gh) = (E_g, B_g)(E_h, B_h)$ and $B_g = B_h$ while $E_g E_h \in E(\Gamma_2)$, so that $h = gs$ and the color $s$ is preserved by the isomorphism. So $\delta$ is an isomorphism as claimed. $\square$

We need more preliminary results before turning to our main result.

**Lemma 3.4.** *Let $G$ be a group and $S \subseteq G$. Let $N \lhd G$, $\mathcal{B}$ be the orbits of $N_L$ and $\mathcal{A}^o$ the group of color-preserving automorphism of* $\mathrm{Cay}(G, S)$*. If $\mathcal{B}$ is an $\mathcal{A}^o$-invariant partition, and $\alpha \in \mathcal{A}^o$, then $\alpha/\mathcal{B}$ is also a color-preserving automorphism of* $\mathrm{Cay}(G/N, S/N)$*, where $S/N = \{sN : s \in S\}$.*

*Proof.* Let $\alpha \in \mathrm{Aut}(\mathrm{Cay}(G,S))$ be a color-preserving automorphism. As $\mathcal{A}^o/\mathcal{B} \leq \mathrm{Aut}(\mathrm{Cay}(G,S)/\mathcal{B})$, we have $\alpha/\mathcal{B} \in \mathrm{Aut}(\mathrm{Cay}(G,S)/\mathcal{B})$. Let $sN \in S/N$ and $g \in G$. As $\alpha$ is a color-preserving automorphism of $\mathrm{Cay}(G,S)$, we have $\alpha(g,gs) = (h,hs)$ or $(h,hs^{-1})$ for some $h \in G$. Then $\alpha(gN,(gN)(sN)) = \alpha(gN,gsN) = (hN,hsN)$ or $(hN,hs^{-1}N)$. Since $hsN = (hN)(sN)$ and $hs^{-1}N = (hN)(s^{-1}N) = (hN)(sN)^{-1}$, we see that $\alpha/\mathcal{B}$ is a color-preserving automorphism of $\mathrm{Cay}(G/N,S/N)$. □

**Lemma 3.5.** *Let $G$ be a group of odd square-free order, and $\mathrm{Cay}(G,S)$ a Cayley graph such that $\mathcal{A}^o$ admits a normal invariant partition $\mathcal{B}$. Then the orbits of $\mathrm{fix}_{G_L}(\mathcal{B})$ are the blocks of $\mathcal{B}$. In particular, if $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ is semiregular, then $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B}) = \mathrm{fix}_{G_L}(\mathcal{B})$.*

*Proof.* Let $k$ be the size of each $B \in \mathcal{B}$, so $k$ is odd and square-free. Suppose to the contrary that for some prime divisor $p$ of $k$, $p$ does not divide $|\mathrm{fix}_{G_L}(\mathcal{B})|$. Now, a Sylow $p$-subgroup $P$ of $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ has order $p^i$ for some $i \geq 1$. Let $P'$ be a Sylow $p$-subgroup of $G_L$. Then $P'$ has order $p$, and by assumption does not lie in $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$. Since $P$ lies in a normal subgroup (namely, $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$) of $\mathcal{A}^o$ that does not contain $P'$, we see that $P$ and $P'$ are not conjugate in $\mathcal{A}^o$. Thus, $P$ and $P'$ cannot both be Sylow $p$-subgroups of $\mathcal{A}^o$. This means that $p^2$ must divide $|\mathcal{A}^o|$. Hence $|\mathrm{Stab}_{\mathcal{A}^o}(1)|$ is divisible by $p$. This implies $p = 2$ (since every point-stabilizer of $\mathcal{A}^o$ is a 2-group by [8, Lemma 6.3]), a contradiction. We conclude that $k$ divides $|\mathrm{fix}_{G_L}(\mathcal{B})|$, so that the orbits of $\mathrm{fix}_{G_L}(\mathcal{B})$ are the blocks of $\mathcal{B}$.

If $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ is semiregular, then $|\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})| \leq k$. Since $\mathrm{fix}_{G_L}(\mathcal{B}) \leq \mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ is semiregular of order $k$, we must have $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B}) = \mathrm{fix}_{G_L}(\mathcal{B})$. □

**Lemma 3.6.** *Let $G$ be a group, and $\Gamma = \mathrm{Cay}(G,S)$ a Cayley graph on $G$. Suppose that $G = G_1 \times G_2$, where $\Gamma_1 = \mathrm{Cay}(G_1,S_1)$ and $\Gamma_2 = \mathrm{Cay}(G_2,S_2)$, for some $S_1 \subset G_1, S_2 \subset G_2$, and that $\Gamma = \Gamma_1 \,\square\, \Gamma_2$. Further assume that $\Gamma_1$ and $\Gamma_2$ have no common factors with respect to Cartesian decomposition. If $\Gamma_1$ and $\Gamma_2$ are both CCA graphs (on $G_1$ and $G_2$ respectively), then $\Gamma$ is a CCA graph on $G$.*

*Proof.* By [9, Corollary 15.6], we have $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2)$. Although their statement assumes that we have a prime factorization (with respect to the Cartesian product), this only affects the outcome if some prime factor is common to $\Gamma_1$ and $\Gamma_2$, which we have assumed is not the case.

If $\alpha$ is an automorphism of $\Gamma$ that fixes $1_G$, then this implies that $\alpha$ can be written as $(\alpha_1,\alpha_2)$, where $\alpha_i \in \mathrm{Aut}(\Gamma_i)$ for $i = 1,2$. Since $\Gamma_i$ is a CCA graph on $G_i$, this implies that $\alpha_i$ is an automorphism of $G_i$. It is then easy to see that $\alpha = (\alpha_1,\alpha_2)$ is an automorphism of $G = G_1 \times G_2$. □

**Lemma 3.7.** *Let $G$ be a group of odd square-free order with subgroups $G_1, G_2 \leq G$ such that $\langle G_1, G_2 \rangle = G$, $G_1 \cap G_2 = 1$, $G_2 \triangleleft G$, and $G_2 \cong F_{21}$. Then $G \cong F_{21} \times G_1$.*

*Proof.* Conjugation of $G_2$ by any element $g$ induces an automorphism $\iota_g$ of $G_2$. We will show that if $g \in G_1$ then $\iota_g = 1$. This will imply that elements of $G_1$ and $G_2$ commute, in which case

$G \cong G_1 \times G_2$ with the other hypothesis. Now, by direct computations or applying [5, Lemma 2.5] and [3, Corollary 4.2B], we see that $\mathrm{Aut}(F_{21})$ has order 42. Also, if $g \in G_1$ then $|\iota_g|$ divides $|g|$, which as $|G|$ is odd and square-free is of relatively prime order to 42. Thus $\iota_g = 1$ as required. $\square$

**Lemma 3.8.** *Let $\Gamma = \mathrm{Cay}(G,S)$ be a Cayley graph of a group $G$ of odd order. Suppose that $\mathcal{A}^o$ admits a normal invariant partition $\mathcal{B}$. Then $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ is faithful in its action on $B \in \mathcal{B}$.*

*Proof.* Let $K$ be the kernel of the action of $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ on $B \in \mathcal{B}$, and $B' \in \mathcal{B}$. As $K$ stabilizes each element of $B$, it must be the case that $K$ is a 2-group by [8, Lemma 6.3], and so $K|_{B'}$ is a normal 2-subgroup of $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})|_{B'}$. The orbits of $K|_{B'}$ then form an invariant partition of $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})|_{B'}$ with blocks of size a power of 2. Since $n$ is odd, this implies that the blocks have size 1, so $K$ is faithful. $\square$

**Lemma 3.9.** *Let $\Gamma = \mathrm{Cay}(G,S)$ for a group $G$ of odd square-free order. Suppose that $\mathcal{A}^o$ has an invariant partition $\mathcal{B}$ formed by the orbits of a normal subgroup $N$ of prime order $p$. If $\gamma \in \mathcal{A}^o$ such that $\gamma/\mathcal{B}$ normalizes $G_L/\mathcal{B}$, then $\gamma$ normalizes $G_L$.*

*Proof.* As $N$ is of prime order and its orbits form $\mathcal{B}$, $N \triangleleft \mathrm{Stab}_{\mathcal{A}^o}(B)$, where $B \in \mathcal{B}$. Then $\mathrm{Stab}_{\mathcal{A}^o}(B)$ in its action on $B$ is permutation isomorphic to a subgroup of $\mathrm{AGL}(1,p)$. It then follows by the Embedding Theorem that $\mathcal{A}^o$ is permutation isomorphic to a subgroup of $\mathcal{A}^o/\mathcal{B} \wr \mathrm{AGL}(1,p)$. We assume without loss of generality that $\mathcal{A}^o \le \mathcal{A}^o/\mathcal{B} \wr \mathrm{AGL}(1,p)$, and so the set on which $\mathcal{A}^o$ is acting is $(G/N) \times \mathbb{Z}_p$. Let $\tau \in \mathcal{A}^o$ so that $\tau(i,j) = (\psi(i), \alpha_i j + b_i)$, where $\psi \in S_{G/N}$, $\alpha_i \in \mathbb{Z}_p^*$, and $b_i \in \mathbb{Z}_p$. By Lemma 3.8, $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ is faithful in its action on $B \in \mathcal{B}$. This then implies that $N$ is a Sylow $p$-subgroup of $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ (as it has order $p$), and we may additionally assume without loss of generality that $N$ is generated by the map $(i,j) \mapsto (i, j+1)$. A straightforward computation using $\gamma^{-1}N\gamma = N$ will show $\alpha_i = \alpha_k$ for every $i,k \in G/N$.

Let $\gamma \in \mathcal{A}^o$ such that $\gamma/\mathcal{B}$ normalizes $G_L/\mathcal{B}$. Let $g_L \in G_L$ with $g_L(i,j) = (\sigma(i), \beta j + c_i)$ where $\sigma \in S_{G/N}$, $\beta \in \mathbb{Z}_p^*$, and $c_i \in \mathbb{Z}_p$, and set $n = |\sigma|$. As $g_L^n(i,j) = (\sigma^n(i), \beta^n j + d_i)$ for suitable $d_i$, and $g_L$ has odd order, we see that $\beta^n$ also has odd order, and as $g_L^n \in \mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$, we see that $|\beta|$ divides $|G|$ (and so is odd). As $\gamma/\mathcal{B}$ normalizes $G_L/\mathcal{B}$, there exists $h \in G$ such that $\gamma^{-1}g_L\gamma/\mathcal{B} = h_L/\mathcal{B}$. Set $\gamma(i,j) = (\delta(i), \alpha j + b_i)$, and $h_L(i,j) = (\theta(i), \mu j + d_i)$, where $\delta, \theta \in S_{G/N}$, $\alpha, \mu \in \mathbb{Z}_p^*$, and $b_i, d_i \in \mathbb{Z}_p$. Computations will show that $\gamma^{-1}g_L\gamma h_L^{-1}(i,j) = (i, \beta\mu^{-1}j + e_i)$ for appropriate $e_i \in \mathbb{Z}_p$, and $\gamma^{-1}g_L\gamma h_L^{-1} \in \mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$. As both $\beta$ and $\mu$ have odd order, $\beta\mu$ has odd order, and as $|\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})| = p \cdot 2^\ell$ for some $\ell \ge 0$ by [8, Lemma 6.3], we see that $\gamma^{-1}g_L\gamma h_L^{-1}$ is in $N \le G_L$, so $\gamma$ normalizes $G_L$. $\square$

The following result is proven as part of [8, Theorem 6.8]. In the proof of that result, Case 1 deals with the possibility that $N$ is elementary abelian, and Case 2 shows that otherwise, $N \cong (\mathrm{PSL}(2,7))^k$, for some $k \ge 1$.

**Lemma 3.10.** *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a connected non-CCA graph of odd order, and let* $N$ *be a minimal normal subgroup of* $\mathcal{A}^o$. *Then* $N$ *is either elementary abelian or* $N \cong (\mathrm{PSL}(2,7))^k$, *for some* $k \geq 1$.

**Lemma 3.11.** *Let* $n$ *be odd and square-free,* $G$ *a group of order* $n$, *and* $\Gamma = \mathrm{Cay}(G, S)$ *be a connected Cayley graph of* $G$. *Suppose that*

- $\mathcal{A}^o$ *has an invariant partition* $\mathcal{B}$ *formed by the orbits of a normal subgroup* $N$ *of prime order* $p$,
- $\mathcal{A}^o/\mathcal{B}$ *does not normalize* $G_L/\mathcal{B}$,
- $G/N = H \times F_{21}$ *where* $|H| = n/(21p)$ *and* $F_{21}$ *is isomorphic to the nonabelian group of order* 21, *and*
- *every non-CCA graph of* $G/N$ *is of the form* $\Gamma_1' \,\square\, \Gamma_2'$ *where* $\Gamma_1'$ *is a CCA-graph of* $H$ *and* $\Gamma_2'$ *is the unique non-CCA graph of* $F_{21}$.

*Then* $\mathcal{A}^o$ *has an invariant partition* $\mathcal{C}$ *consisting of* $n/(21p)$ *blocks of size* $21p$ *and* $\mathrm{fix}_{\mathcal{A}^o/\mathcal{B}}(\mathcal{C}/\mathcal{B}) = \mathrm{PSL}(2,7)$.

*Proof.* As $G_L/\mathcal{B}$ is not normal in $\mathcal{A}^o/\mathcal{B}$ and $\mathcal{A}^o/\mathcal{B}$ is contained in the color-preserving group of automorphisms $A$ of $\Gamma' = \mathrm{Cay}(G/N, S/N) = \mathrm{Cay}(G,S)/N$ by Lemma 3.4, $\Gamma'$ is not CCA. By hypothesis $\Gamma' = \Gamma_1' \,\square\, \Gamma_2'$ where $\Gamma_1'$ is a CCA-graph of a group $H$ of order $n/(21p)$ and $\Gamma_2'$ is the unique non-CCA graph of $F_{21}$. Then $A$ admits an invariant partition $\mathcal{C}/\mathcal{B}$ consisting of blocks of size 21. As $\mathcal{A}^o/\mathcal{B} \leq A$, this then implies that $\mathcal{A}^o$ admits an invariant partition $\mathcal{C}$ with blocks of size $21p$. The induced subgraph on each block of $\mathcal{C}/\mathcal{B}$ consists of a copy of $\Gamma_2'$.

Now, $\mathrm{Aut}(\Gamma') \leq \mathrm{Aut}(\Gamma_1') \times \mathrm{Aut}(\Gamma_2')$ since $\Gamma' = \Gamma_1' \square \Gamma_2'$ and $\gcd(|\Gamma_1'|, |\Gamma_2'|) = 1$. Thus $A \leq A_1 \times A_2$, where $A_i$ is the color-preserving group of automorphisms of $\Gamma_i'$. Also, $A_1 \leq H \rtimes \mathrm{Aut}(H)$ since $\Gamma_1'$ is CCA on $H$, and $A_2 \leq \mathrm{PSL}(2,7)$ by Proposition 2.6. So $\mathcal{A}^o/\mathcal{B} \leq A \leq (H \rtimes \mathrm{Aut}(H)) \times \mathrm{PSL}(2,7)$. Also, since $\mathcal{A}^o/\mathcal{B}$ does not normalize $G_L/\mathcal{B} = G/N = H \times F_{21}$, there must be some element of $\mathcal{A}^o/\mathcal{B}$ that is not in $\mathrm{Aut}(H) \times F_{21}$. Since $F_{21}$ is a maximal subgroup of $\mathrm{PSL}(2,7)$ [10], the result follows. □

The following result is the main result of this paper.

**Theorem 3.12.** *Let* $n$ *be odd and square-free,* $G$ *a group of order* $n$, *and* $\Gamma = \mathrm{Cay}(G, S)$ *be a connected Cayley graph of* $G$. *Then* $\Gamma$ *is a non-CCA Cayley graph of* $G$ *if and only if*

(1) $n$ *is divisible by* 21 *and* $G = G_1 \times F_{21}$, *where* $G_1$ *is a group of order* $n/21$ *and* $F_{21}$ *is the nonabelian group of order* 21, *and*

(2) $\Gamma = \Gamma_1 \,\square\, \Gamma_2$, *where* $\Gamma_1$ *is a CCA graph of order* $n/21$ *and* $\Gamma_2$ *is the unique non-CCA graph of order* 21.

*Proof.* Suppose that $G$ and $\Gamma$ satisfy (1) and (2). Then $\Gamma$ is not CCA by [8, Proposition 3.1].

Conversely, we proceed by induction on $n$, with the base case being $n = 21$ as by [8, Theorem 6.8] that is the smallest positive odd square-free integer for which there is a non-CCA Cayley graph of some group $G$, and that group is $G = F_{21}$. By Proposition 2.6, there is a unique non-CCA Cayley digraph of $F_{21}$. Let $n > 21$ and assume that the result holds for all $21 \leq m < n$. Let $G$ be a group of order $n$, and $\Gamma = \mathrm{Cay}(G, S)$ be connected.

Let $N$ be a minimal normal subgroup of $\mathcal{A}^o$, so that by Lemma 3.10 either $N$ is an elementary abelian $p$-group for some prime $p|n$, or $N = (\mathrm{PSL}(2,7))^k$ for some $k \geq 1$. The orbits of $N$ form an invariant partition $\mathcal{B}$ of $\mathcal{A}^o$.

Suppose $\mathcal{B} = \{G\}$ is a trivial invariant partition. If $N$ is an elementary abelian $p$-group, then as $n$ is square-free we have $n = p$ is prime. Now by Burnside's Theorem [3, Theorem 3.5B], $\mathcal{A}^o \leq \mathrm{AGL}(1,p)$ so every element of $\mathcal{A}^o$ is affine, contradicting our assumption that $\Gamma$ is not CCA. Otherwise, as $n$ is square-free, the O'Nan Scott Theorem implies that $\mathrm{soc}(\mathcal{A}^o)$ is a simple group (see for example [6, Lemma 2.1]) and so $N = \mathrm{PSL}(2,7)$. Additionally, by [8, Lemma 6.3] $\mathrm{Stab}_{\mathrm{soc}(\mathcal{A}^o)}(v)$ is a 2-group for any vertex $v$, and $\mathrm{Stab}_{\mathrm{soc}(\mathcal{A}^o)}(v) = D_8$, which is a Sylow 2-subgroup of $\mathrm{PSL}(2,7)$. This then implies that $n = 21$, contradicting $n > 21$.

Suppose $\mathcal{B} \neq \{G\}$, then remembering that $n$ is odd and applying [6, Theorem 2.10], it follows that $N$ in its action on $B \in \mathcal{B}$, written $N|_B$, is either a simple group $T$ or $A_7^2$ of degree 105. As $\mathrm{Stab}_{\mathcal{A}^o}(v)$ is a 2-group for every $v \in G$ by [8, Lemma 6.3] we see that $N|_B = \mathbb{Z}_p$ or $\mathrm{PSL}(2,7)$. By Lemma 3.8, we have the action of $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ on $B \in \mathcal{B}$ is faithful, and so $N = \mathbb{Z}_p$ or $\mathrm{PSL}(2,7)$.

First we suppose $N = \mathrm{PSL}(2,7)$. Let $k$ be the size of the blocks of $\mathcal{B}$, and note that $k$ is odd. Then $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})|_B$ has order $k \cdot 2^\ell$ for some $\ell$, and so the stabilizer of a point in $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})|_B$ is a Sylow 2-subgroup of $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})|_B$ for every $B \in \mathcal{B}$. Additionally, as any two point stabilizers of a group are conjugate and Sylow 2-subgroups are certainly conjugate, the stabilizer of a point in $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ of $b \in B$ is the stabilizer of a point in $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ of $b' \in B'$, where $B, B' \in \mathcal{B}$. We conclude that the stabilizer of $b \in B$ in $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ fixes at least one point in every block of $\mathcal{B}$. Additionally, the stabilizer of $b \in B$ in $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ will fix exactly one point in every block of $\mathcal{B}$ provided that the stabilizer of a point $b \in B$ of $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ in its action on $B$ fixes exactly one point. We claim that this occurs.

Note that $\mathrm{PSL}(2,7)$ has order 168. Since $n$ is odd and square-free, this forces each block $B$ to have length dividing 21. The action of $N|_B$ can only be imprimitive if the block $B$ has length 21. In this case, the stabilizer of a point in $\mathrm{PSL}(2,7)$ under its action on $B \in \mathcal{B}$ fixes exactly one point by the ATLAS of Finite Group Representations [10], since the lengths of the suborbits are 1, 2, 2, 4, 4, and 8. If $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B}) = \mathrm{PSL}(2,7)$, then our claim follows, and otherwise the claim also follows as we would then have that $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ is isomorphic to one of $\mathrm{PGL}(2,7)$ or $\mathrm{P\Gamma L}(2,7)$, each of which is primitive in its action on $B \in \mathcal{B}$, by the ATLAS of Finite Group Representations [10].

Now, by Lemma 3.5 we have that $\mathcal{B}$ is formed by the orbits of $\mathrm{fix}_{G_L}(\mathcal{B})$ which is normal in $G_L$. As the only regular subgroup of $\mathrm{PSL}(2,7)$ in its action on 21 points is $F_{21}$, setting $(\hat{G}_2)_L = \mathrm{fix}_{G_L}(\mathcal{B})$ forces $G_2 \cong F_{21}$. Let $S_1 = S \setminus G_2$. As the stabilizer of $b \in B$ in $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{B})$ fixes exactly one point in every block of $\mathcal{B}$, we see that $G_1 = \langle S_1 \rangle$ is an equivalence class of the equivalence relation $\equiv$

as defined in Lemma 3.6, and $|G_1| = n/21$. We know $F_{21} \cong G_2 \triangleleft G$, and order arguments give $G_1 \cap G_2 = 1$ and $\langle G_1, G_2 \rangle = G$. By Lemma 3.7 we have $G \cong G_1 \times G_2$.

We may now apply Lemma 3.3 and conclude $\Gamma = \Gamma_1 \,\Box\, \Gamma_2$, where $\Gamma_i = \mathrm{Cay}(G_i, S_i)$ for $i = 1, 2$. By Lemma 3.6, there must be some $i \in \{1, 2\}$ such that $\Gamma_i$ is not a CCA graph on $G_i$. As $\Gamma_1$ has odd square-free order less than $n$ and relatively prime to 21, our inductive hypothesis implies that it is CCA, so $\Gamma_2$ (being a Cayley graph on the nonabelian group of order 21) is isomorphic to the unique non-CCA graph of order 21, and we are done.

It now only remains to consider the case where $N$ is of order $p$. If $G_L/\mathcal{B} \triangleleft \mathcal{A}^o/\mathcal{B}$, then as $G_L$ is transitive we have $G_L \triangleleft \mathcal{A}^o$ by Lemma 3.9. Then $\Gamma$ is CCA (see [8, Remark 6.2]). We conclude that $G_L/\mathcal{B}$ is not normal in $\mathcal{A}^o/\mathcal{B}$ and $\mathcal{A}^o/\mathcal{B}$ is contained in the color-preserving group of automorphisms $A$ of $\mathrm{Cay}(G/N, S/N) = \mathrm{Cay}(G, S)/N$ by Lemma 3.4. Then $\mathrm{Cay}(G/N, S/N)$ is not CCA, and so by the induction hypothesis $G/N = H \times F_{21}$ and $\mathrm{Cay}(G/N, S/N) = \Gamma_1' \,\Box\, \Gamma_2'$ where $\Gamma_1'$ is a CCA-graph of $H$ of order $n/(21p)$ and $\Gamma_2'$ is the unique non-CCA graph of $F_{21}$. By Lemma 3.11 $\mathcal{A}^o$ has an invariant partition $\mathcal{C}$ consisting of $n/(21p)$ blocks of size $21p$ and $\mathrm{fix}_{\mathcal{A}^o/\mathcal{B}}(\mathcal{C}/\mathcal{B}) = \mathrm{PSL}(2, 7)$.

By Lemma 3.8, we have the action of $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{C})$ on $C \in \mathcal{C}$ is faithful. Now, as conjugation by an element of $\mathcal{A}^o$ induces an automorphism of $N$, there is a homomorphism $\phi : \mathcal{A}^o \mapsto \mathrm{Aut}(\mathbb{Z}_p)$, and of course $\mathrm{Aut}(\mathbb{Z}_p)$ is cyclic of order $p - 1$. As $H \cong G_L/\mathcal{C} \triangleleft \mathcal{A}^o/\mathcal{C}$ (because $\Gamma_1'$ is CCA) and $G_L$ is metacyclic as $n$ is square-free [7, Chapter 7, Theorem 6.2], $G_L/\mathcal{C}$ has a solvable automorphism group, and so $\mathcal{A}^o/\mathcal{C}$ is solvable. Hence any minimal subgroup of $\mathcal{A}^o$ with a section isomorphic to $\mathrm{PSL}(2, 7)$ is contained in $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{C})$. Let $K = \mathrm{Ker}(\phi) \cap \mathrm{fix}_{\mathcal{A}^o}(\mathcal{C}) \triangleleft \mathcal{A}^o$.

Now, $N \leq \mathrm{Ker}(\phi)$ and as $\mathrm{fix}_{\mathcal{A}^o}(\mathcal{C})/\mathcal{B} \cong \mathrm{fix}_{\mathcal{A}^o/\mathcal{B}}(\mathcal{C}/\mathcal{B}) = \mathrm{PSL}(2, 7)$ is simple (and so in particular is not cyclic), we see that $\mathrm{fix}_{\mathrm{Ker}(\phi)}(\mathcal{C})/\mathcal{B} = \mathrm{PSL}(2, 7)$. Observe that $K = \mathrm{fix}_{\mathrm{Ker}(\phi)}(\mathcal{C})/\mathcal{B}$, so that $K/\mathcal{B} \cong \mathrm{PSL}(2, 7)$. Also, by the definition of $\phi$, $N$ is central in $\mathrm{Ker}(\phi)$, and so in $K$. Since no element of $\mathrm{Aut}(N)$ other than the elements of $N$ themselves will be in $\mathrm{Ker}(\phi)$, we have $\mathrm{fix}_K(\mathcal{B}) = N$, so $K/\mathcal{B} = K/N$. This implies that $K$ is a central extension of the perfect group $\mathrm{PSL}(2, 7)$. If the commutator subgroup $L$ of $K$ is not equal to $K$, then as $K/\mathcal{B} \cong \mathrm{PSL}(2, 7)$ and $|N| = p$, we have that $L \cong \mathrm{PSL}(2, 7)$. Then $L$ is a normal subgroup of $\mathcal{A}^o$, and so $\mathcal{A}^o$ admits a normal invariant partition $\mathcal{D} \prec \mathcal{C}$ with blocks of size 21. Then $\mathcal{A}^o$ contains a minimal normal subgroup that is not a cyclic group of prime degree, and this case reduces to one considered above. Otherwise, $L = K$ is a perfect central extension of $\mathrm{PSL}(2, 7)$, and as $\mathrm{PSL}(2, 7)$ has Schur multiplier 2, by [1, Theorem 33.8] we have $p = 2$, a contradiction. $\square$

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