Automorphism groups with cyclic commutator subgroup and Hamilton cycles

Edward Dobson
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803
U.S.A.

Heather Gavlas
Department of Mathematics and Statistics
Grand Valley State University
Allendale, MI 49401
U.S.A.

Joy Morris
Department of Mathematics and Statistics
Simon Fraser University
Burnaby, BC V5A 1S6
Canada

Dave Witte
Department of Mathematics
Oklahoma State University
Stillwater, OK 74078
U.S.A.

Abstract

It has been shown that there is a Hamilton cycle in every connected Cayley graph on any group G whose commutator subgroup is cyclic of prime-power order. This note considers connected, vertex-transitive graphs X of order at least 3, such that the automorphism group of X contains a vertex-transitive subgroup G whose commutator subgroup is cyclic of prime-power order. We show that of these graphs, only the Petersen graph is not hamiltonian.

1. Introduction

Considerable attention has been devoted to the problem of determining whether or not a connected, vertex-transitive graph X has a Hamilton cycle [A1], [WG]. The vertex-transitivity implies that some group G of automorphisms of X acts transitively on $V(X)$. If G can be chosen to be abelian, it is easy to see that X has a Hamilton cycle, so it is natural to try to prove the same conclusion when G is "almost abelian." Thus, recalling that the *commutator subgroup* of G is the subgroup $G' = \langle x^{-1}y^{-1}xy : x, y \in G \rangle$, and that G is abelian if and only if the commutator subgroup of G is trivial, it is natural to consider the case where the commutator subgroup of G is "small" in some sense. In this vein, K. Keating and D. Witte [KW] used a method of D. Marušič [M] to show that there is a Hamilton cycle in every Cayley graph on any group whose commutator subgroup is cyclic of prime-power order. This note utilizes techniques of B. Alspach, E. Durnberger, and T. Parsons [AP, ADP, A2] to extend this result to vertex-transitive graphs.

Theorem 1.1. *Let X be a connected vertex-transitive graph of order at least 3. If there is a vertex-transitive group G of automorphisms of X, such that the commutator subgroup of G is cyclic of prime-power order, then X is the Petersen graph or X is hamiltonian.*

Because K_2 and the Petersen graph have Hamilton paths, the following corollary is immediate.

Corollary 1.2. *Let X be a connected vertex-transitive graph. If there is a vertex-transitive group G of automorphisms of X, such that the commutator subgroup of G is cyclic of prime-power order, then X has a Hamilton path.*

Acknowledgment. Much of this research was carried out at the Centre de Recherches Mathématiques of the Université de Montréal. The authors would like to thank the organizers and participants of the Workshop on Graph Symmetry, and the staff of the CRM, for the stimulating environment they provided. Witte was partially supported by a grant from the National Science Foundation.

2. Assumptions and definitions

Assumption 2.1. *Throughout this note, X denotes a connected, vertex-transitive graph, G is a group of automorphisms of X that acts transitively on the vertex set $V(X)$, and G' is the commutator subgroup of G. Furthermore, we assume G' is cyclic of order p^k, where p is a prime, and that X has at least three vertices.*

Assumption 2.2. *We also assume X is G-minimal. That is, if Y is a connected, spanning subgraph of X, such that, for all $g \in G$, we have $gY = Y$, then it must be the case that $Y = X$. This causes no loss of generality, because a Hamilton cycle in any such subgraph Y would also be a Hamilton cycle in X, so there would be no harm in replacing X with Y.*

Definition 2.3 (cf. [Sc, p. 255])**.** *The stabilizer G_x of a vertex $x \in V(X)$ is $\{g \in G : g(x) = x\}$. This is a subgroup of G.*

Lemma 2.4 [Sc, 10.1.2, p. 256]**.** *Let $x \in V(G)$ and $g \in G$. Then $G_{gx} = g(G_x)g^{-1}$.*

Corollary 2.5. *If H is a normal subgroup of G, then the following are equivalent:*
- *a) HG_x is a normal subgroup of G, for some $x \in V(X)$;*
- *b) HG_x is a normal subgroup of G, for every $x \in V(X)$;*
- *c) $HG_x = HG_y$, for all $x, y \in V(X)$.*

Corollary 2.6. *For every $x \in V(X)$, the stabilizer G_x does not contain any nontrivial, normal subgroup of G.*

Proof. Let H be a normal subgroup of G that is contained in G_x. Lemma 2.4 implies $H \subset G_{gx}$, for all $g \in G$. Because G is vertex-transitive, this means $H \subset G_y$, for all $y \in V(X)$. Therefore, the identity automorphism of X is the only element of H. □

Definition 2.7 [Sc, p. 255]**.** *Let H be a subgroup of G, and let $x \in V(X)$. The H-orbit of x is $\{hx : h \in H\}$. (The H-orbits form a partition of $V(X)$.) Note that if H is normal in G, then the subgraphs of X induced by different H-orbits are isomorphic, because $g(Hx) = H(gx)$ in this case.*

Definition 2.8. *Let H be a subgroup of G. The quotient graph X/H is that graph whose vertices are the H-orbits, and two such vertices Hx and Hy are adjacent in X/H if and only if there is an edge in X joining a vertex of Hx to a vertex of Hy. Note that if H is normal in G, then X/H is vertex-transitive: the action of G on $V(X)$ factors through to a transitive action of G/H on $V(X/H)$, by automorphisms of X/H.*

Lemma 2.9. *If H is a normal subgroup of G, then every path in X/H lifts to a path in X.*

Proof. It suffices to show that if Hx is adjacent to Hy in X/H, then x is adjacent to some vertex in Hy. By definition of X/H, we know that some $\tilde{x} \in Hx$ is adjacent to some $\tilde{y} \in Hy$. There exists $h \in H$ with $x = h\tilde{x}$, so x is adjacent to $h\tilde{y} \in Hy$. □

Definition 2.10 [B, Defn. 16.1, p. 123]**.** *Let S be a subset of G, and assume $s^{-1} \in S$, for all $s \in S$. The Cayley graph* $\mathrm{Cay}(G;S)$ *is the graph whose vertices are the elements of G, and such that there is an edge from g to h iff $gs = h$, for some $s \in S$. It is clear that G acts transitively on the vertices of* $\mathrm{Cay}(G;S)$ *by left multiplication, so* $\mathrm{Cay}(G;S)$ *is vertex transitive. The Cayley graph is connected iff S generates G, in which case, it is G-minimal iff no proper, symmetric subset of S generates G.*

Recall that G' is a normal subgroup of G and the quotient group G/G' is abelian [Sc, Thms. 3.4.11 and 3.4.10, p. 59]. Since G/G' is abelian and transitive on $V(X/G')$, it follows from the following basic fact that X/G' is a Cayley graph on the abelian group $G/(G_xG')$, for any $x \in V(X)$.

Lemma 2.11 (Sabidussi [Sa], [B, Lem. 16.3, p. 124])**.** *If $G_x = e$, for some $x \in V(X)$, then X is (isomorphic to) a Cayley graph on G.*

3. Preliminaries on the Frattini subgroup

Assumptions 2.1 and 2.2 are in effect. The main result of this section is Lemma 3.6.

Definition 3.1 [Sc, 7.3.1 and 7.3.2, p. 159]**.** *An element g of G is a nongenerator if, for every subset S of G, such that $\langle S, x\rangle = G$, we have $\langle S\rangle = G$. The Frattini subgroup of G, denoted $\Phi(G)$, is the set of all nongenerators of G; it is a subgroup of G.*

Lemma 3.2. *If H is any subgroup of G', then H is normal in G, and $H^p \subset \Phi(G)$, where $H^p = \langle h^p : h \in H\rangle$.*

Proof. Because G' is a cyclic normal subgroup of G, we know that every subgroup of G' is a normal subgroup of G [G, Thm. 1.3.1(i), p. 9, and Thm. 2.1.2(ii), p. 16]. Therefore H is normal in G, so $\Phi(H) \subset \Phi(G)$ [Sc, 7.3.17, p. 162]. Because H is a cyclic p-group, it is not difficult to see that $\Phi(H) = H^p$ [Sc, 7.3.7, p. 160]. □

Lemma 3.3. *If H is a normal subgroup of G, and $H \subset \Phi(G)$, then X/H is G-minimal.*

Proof. Let Y be a connected, spanning subgraph of X/H, such that, for all $g \in G$, we have $gY = Y$. Choose $x \in V(X)$, and let

$$S = \{s \in G : sx \text{ is adjacent to } x \text{ in } X\}, \text{ and}$$

$$T = \{t \in G : Htx \text{ is adjacent to } Hx \text{ in } X/H\}.$$

It is straightforward to verify that $G_xSG_x = S$ and $HG_xTG_x = T$. Furthermore, because Y is connected, we see that T generates G.

Since Y is a subgraph of X/H, we must have $T \subset HS$, so, because $HT = T$, this implies $T = H(S \cap T)$. Then, because T generates G, and $H \subset \Phi(G)$, we conclude that $S \cap T$ generates G. Therefore, letting Z be the spanning subgraph of X whose edge set is

$$E(Z) = \{[gtx, gx] \mid g \in G,\ t \in S \cap T\},$$

we see that Z is connected. So the G-minimality of X implies that $S \cap T = S$. Therefore $HS = H(S \cap T) = T$, so $X/H = Y$. □

Because a G-minimal graph has no loops, we have the following corollary.

Corollary 3.4. *If H is a normal subgroup of G, and $H \subset \Phi(G)$, then the subgraph of X induced by each H-orbit has no edges.*

We now recall (in a weak form) the fundamental work of C. C. Chen and N. F. Quimpo [CQ].

Theorem 3.5 (Chen-Quimpo [CQ])**.** *Let Y be a connected Cayley graph on an abelian group of order at least three. Then each edge of Y (except any loop) is contained in some Hamilton cycle of Y.*

The following helpful result is the main conclusion obtained from our discussion of G-minimality and Frattini subgroups. (It also relies on the Chen-Quimpo Theorem.)

Lemma 3.6. *If H is a subgroup of G', such that X/H has a Hamilton cycle, then each edge of X/H (except any loop) is contained in some Hamilton cycle of X/H.*

Proof. If $H = G'$, then G/H is abelian, so the desired conclusion follows from the Chen-Quimpo Theorem (3.5).

We may now assume $H \neq G'$, which implies $H \subset (G')^p$. So $H \subset \Phi(G)$ (see 3.2); therefore X/H is G-minimal (see 3.3). Let C be any Hamilton cycle in X/H, and let $Y = \cup_{g \in G} gC$. Because X/H is G-minimal, we must have $Y = X/H$, so every edge of X/H is contained in some Hamilton cycle gC. □

4. Proof of Theorem 1.1

Assumptions 2.1 and 2.2 are in effect. The main conclusions of this section are the two propositions. Together, they constitute a proof of Theorem 1.1.

Let us begin by disposing of a trivial case: suppose that X/G' has only one vertex. Then G' is transitive on $V(X)$, so there is no harm in replacing G with G'; hence G is cyclic and thus G is abelian. So Theorem 3.5 implies that X has a Hamilton cycle unless X has less than three vertices.

Lemma 4.1. *Suppose H is a subgroup of G'. If there is a path*

$$H^p x_1, H^p x_2, \ldots, H^p x_n, H^p x_{n+1}$$

in X/H^p, with $H^p x_1 \neq H^p x_{n+1}$, such that the image $Hx_1, Hx_2, \ldots, Hx_n, Hx_{n+1}$ of this path in X/H is a Hamilton cycle (or, if $X/H \cong K_2$, such that $n = 2$ and $Hx_1 = Hx_3 \neq Hx_2$), then X has a Hamilton cycle.

Proof. We can lift the path $H^p x_1, H^p x_2, \ldots, H^p x_n, H^p x_{n+1}$ to a path in X (see 2.9), so we may assume $x_1, x_2, \ldots, x_{n+1}$ is a path in X. Because $Hx_1 = Hx_{n+1}$, there exists $\gamma \in H$ such that $\gamma(x_1) = x_{n+1}$. Now, because $x_{n+1} \notin H^p x_1$, we know $\gamma \notin H^p$, which implies that γ generates H. Let P be the path $x_1, x_2, \ldots, x_n$. Then the trail $P, \gamma(P), \ldots \gamma^{|H|-1}(P), x_1$ is a Hamilton cycle in X. □

The analysis now breaks into two cases, depending on whether or not the subgraphs induced by each G'-orbit are empty. Note that all of these subgraphs are isomorphic (because G' is a normal subgroup), so either all are empty, or none are.

Proposition 4.2. *If the subgraph induced by each G'-orbit has no edges, then X has a Hamilton cycle.*

Proof (cf. [AP, ADP, A2]). Let $x_1 \in V(X)$. Because G/G' is abelian, we know that $G'G_{x_1}$ is a normal subgroup of G. Hence, there is a subgroup H of G', such that HG_{x_1} is normal in G, but KG_{x_1} is *not* normal in G, for any proper subgroup K of H. (It may be the case that $H = G'$ or $H = e$.) Since X/H is a connected Cayley graph on the group $G/(HG_x)$ (see 2.11), and the commutator subgroup of any quotient of G is cyclic, we know that X/H has a Hamilton cycle (or $X/H \cong K_2$) [KW].

We may assume $H \neq e$, for otherwise $X = X/H$ has a Hamilton cycle, so we are done. Then $H^p \neq H$, so the choice of H implies $H^p G_{x_1}$ is not normal. Therefore, since X is connected and vertex-transitive, it follows from Cor. 2.5 that x_1 is adjacent to some vertex u, such that $H^p G_{x_1} \neq H^p G_u$, which implies that there exists $\gamma \in G_{x_1}$ such that $\gamma(u) \notin H^p u$. However, because $HG_{x_1} = HG_u$ (see 2.5), we have $\gamma(u) \in G_{x_1} u \subset HG_u u = Hu$.

Since the subgraph induced by Hx_1 is contained in the subgraph induced by $G'x_1$, which has no edges, and x_1 is adjacent to u, it follows that $u \notin Hx_1$, and thus the edge $[Hx_1, Hu]$ is not a loop in X/H. Therefore, there exists a Hamilton path from Hx_1 to Hu in X/H (see 3.6). This lifts to an n-path $x_1, x_2, x_3, \ldots, x_n$, where $x_n \in Hu$ (see 2.9). Because not both of

$$H^p u, H^p x_1, H^p x_2, \ldots, H^p x_n \qquad \text{and} \qquad H^p \gamma(u), H^p x_1, H^p x_2, \ldots, H^p x_n$$

can be a cycle, Lemma 4.1 implies there is a Hamilton cycle in X, as desired. □

We now consider the case where the G'-orbits do not induce empty graphs. Let us begin with some preliminary observations.

Lemma 4.3. *If the subgraph induced by each G'-orbit has some edges, then these subgraphs are connected, and p is odd.*

Proof. Suppose the subgraph induced by $G'x$ is not connected. Because G' is cyclic, this subgraph is circulant, so any connected component must be induced by the orbit of some proper subgroup H of G'. But $H \subset (G')^p$, and $(G')^p \subset \Phi(G)$ (see 3.2), so Corollary 3.4 asserts that the subgraph induced by any H-orbit has no edges. This contradicts the fact that the connected components of the subgraph induced by $G'x$ *do* have edges.

We now show p is odd; suppose, to the contrary, that $p = 2$. Let $\bar{G} = G/(G')^2$. The commutator subgroup of $\bar{G}$ is $G'/(G')^2$, which has order 2. Because a group of order 2 has no nontrivial automorphisms, this implies that the commutator subgroup of $\bar{G}$ is contained in the center of $\bar{G}$; therefore $\bar{G}$ is nilpotent (of class 2) [G, p. 21]. Because $(G')^2 \subset \Phi(G)$ (see 3.2), this implies that $G/\Phi(G)$ is nilpotent. Hence G itself is nilpotent [Sc, 7.4.10, p. 168], so $G' \subset \Phi(G)$ [Sc, Thm. 7.3.4, p. 160]. Therefore the subgraph induced by each G'-orbit has no edges (see 3.4). This contradicts our hypothesis. □

We can now concisely state several important results of B. Alspach [A2, A3].

Theorem 4.4 (Alspach)**.** *Assume the subgraph induced by each G'-orbit has some edges. Then X has a Hamilton cycle if any of the following are true:*

a) the subgraph induced by a G'-orbit does not have valence two [A3, Thm. 2.4]*; or*

b) X/G' has only two vertices, and X is not the Petersen graph [A2, Thm. 2]*; or*

c) the number of vertices of X/G' is odd [A3, Thm. 3.7(ii)]*; or* *d) there is a Hamilton cycle in X/G' that can be lifted to a cycle in X* [A3, Thm. 3.9].

Lemma 4.5. *Let $x \in V(X)$. If $G_x = G_y$ for all $y \in G'x$, then X has a Hamilton cycle.*

Proof. This is essentially the same as the proof of Proposition 4.2; the assumption that the subgraph induced by $G'x$ has no edges was used only to show that $u \notin Hx_1$, and this follows from the assumption that $G_x = G_y$ for all $y \in G'x$ (and, hence, for all $y \in Hx$). □

The following lemma shows that we may assume all the vertices in each G'-orbit have different stabilizers. The proof is mainly group-theoretic. The key is the observation that the automorphism group of a cycle is a dihedral group. Therefore, if a group of automorphisms acts transitively on the vertices of an odd cycle, then either all vertices have different stabilizers or all vertices have the same stabilizer, depending on whether or not the group contains any reflections.

Lemma 4.6. *Assume the subgraph induced by each G'-orbit has some edges, and that there are two vertices x and y in the same G'-orbit, such that $G_x = G_y$. Then X has a Hamilton cycle.*

Proof. Let Y be the subgraph of X induced by $G'x$, and let $K = \cap_{v \in G'x} G_v$. (Note that K is a subgroup.) Because every subgroup of G' is normal in G (see 3.2), we know $G' \cap G_x = e$ (see 2.6), so $G' \cap K = e$. On the other hand, since G' fixes $V(Y)$ setwise, we see that G' normalizes K. Therefore, $[G', K] \subset G' \cap K$, so G' must centralize K.

¿From 4.4(a), we can (and do) assume that Y has valence two. Because Y is connected (see 4.3), this means that Y is a cycle. (Because $p \neq 2$ (see 4.3), it is an odd cycle.) Therefore, we see that K is a subgroup of index at most two in G_v, for each $v \in V(Y)$. In fact, from Lemma 4.5, we may assume that the index is exactly two.

Let A be a subgroup of G_x of order two. Because A is not normal in G (see 2.6), we know that A does not centralize G' (otherwise, it would be the only Sylow 2-subgroup of the normal subgroup AG', so it would be normal in G). Because G' is a cyclic p-group (and p is odd), the automorphism group of G' is cyclic [Sc, 5.7.12, p. 120], so it has only one element of order 2, namely, inversion. Therefore, the action of A by conjugation inverts G'. Because G' has odd order, this means that e is the only element of G' that is centralized by A.

On the other hand, A must centralize K (because $A \subset G_x$ normalizes K, and $K \cap G' = e$). Thus, we see that K is the centralizer of AG' in KG'. Because AG' and KG' are normal, this implies that K is a normal subgroup of G. Therefore, $K = e$ (see 2.6), so $G_x = A$ has order 2. Hence, because a group of order 2 has no nontrivial automorphisms, any element of G that normalizes G_x must actually centralize it. In particular, then the conclusion of the preceding paragraph implies that no nontrivial element of G' normalizes G_x. This contradicts the fact that $G_x = G_y$ (see 2.4). □

Proposition 4.7. *If the subgraph induced by each G'-orbit has some edges, then X has a Hamilton cycle, unless X is the Petersen graph.*

Proof (cf. pf. of 4.2). Let H be the smallest subgroup of G', such that, whenever x and y are two adjacent vertices of X that are not in the same G'-orbit, we have $HG_x = HG_y$. (It may be the case that $H = G'$.) Note that, from 4.4(b), we may assume X/G' has more than two vertices.

Assume for the moment that H is nontrivial. Then H^p is properly contained in H, so the minimality of H implies there are two adjacent vertices x_1 and u, such that $G'x_1 \neq G'u$, and $H^pG_{x_1} \neq H^pG_u$. Thus, there exists $\gamma \in G_{x_1}$ such that $\gamma(u) \notin H^pu$. Because X/G' has more than two vertices, we see that X/H is not the Petersen graph, so, from Lemma 3.6 (and induction on the number of vertices in X), we know there is a Hamilton path from Hx_1 to Hu in X/H. This lifts to an n-path $x_1, x_2, \ldots, x_n$, where $x_n \in Hu$ (see 2.9). Because not both of

$$H^pu, H^px_1, H^px_2, \ldots, H^px_n \qquad \text{and} \qquad H^p\gamma(u), H^px_1, H^px_2, \ldots, H^px_n$$

can be a cycle, Lemma 4.1 implies there is a Hamilton cycle in X, as desired.

We may now assume $H = e$. Let $x_1, x_2, \ldots, x_{m+1}$ be a lift to X of some Hamilton cycle in X/G'. Because $H = e$, we must have $G_{x_i} = G_{x_{i+1}}$ for every i, so $G_{x_1} = G_{x_{m+1}}$. Therefore, if $x_1 \neq x_{m+1}$, then Lemma 4.6 implies that X has a Hamilton cycle. On the other hand, if $x_1 = x_{m+1}$, then Theorem 4.4(d) yields the same conclusion. □

References

[A1] B. Alspach, The search for long paths and cycles in vertex-transitive graphs and digraphs, in: *Combinatorial Mathematics VIII*, ed. K.L. McAvaney, Lecture Notes in Mathematics, Vol. 884 (Springer-Verlag, Berlin, 1981) 14–22.

[A2] B. Alspach, Hamilton cycles in metacirculant graphs with prime power cardinal blocks, *Ann. Discrete Math* **41** (1989) 7–16.

[A3] B. Alspach, Lifting Hamilton cycles of quotient graphs, *Discrete Math.* **78** (1989) 25–36.

[ADP] B. Alspach, E. Durnberger, and T. Parsons, Hamilton cycles in metacirculant graphs with prime cardinality blocks, *Ann. Discrete Math* **27** (1985) 27–34.

[AP] B. Alspach and T. Parsons, On hamiltonian cycles in metacirculant graphs, *Ann. Discrete Math* **15** (1982) 1–7.

[B] N. Biggs, *Algebraic Graph Theory, 2nd ed.*, Cambridge Univ. Press, Cambridge, 1993.

[CQ] C.C. Chen and N.F. Quimpo, On strongly hamiltonian abelian group graphs, in: *Combinatorial Mathematics VIII*, ed. K.L. McAvaney, Lecture Notes in Mathematics, Vol. 884 (Springer-Verlag, Berlin, 1981) 23–34.

[G] D. Gorenstein, *Finite Groups*, Chelsea, New York, 1980.

[KW] K. Keating and D. Witte, On Hamilton cycles in Cayley graphs in groups with cyclic commutator subgroup, *Ann. Discrete Math* **27** (1985) 89–102.

[M] D. Marušič, Hamiltonian circuits in Cayley graphs, *Discrete Math* **46** (1983) 49–54.

[Sa] G. Sabidussi, Vertex-transitive graphs, *Monatshefte fur Math.*, **68** (1964) 426–438.

[Sc] W. R. Scott, *Group Theory*, Dover, New York, 1987.

[WG] D. Witte and J. A. Gallian, A survey: hamiltonian cycles in Cayley digraphs, *Discrete Math.*, **51** (1984) 293–304.