CYCLIC HAMILTONIAN CYCLE SYSTEMS OF THE COMPLETE GRAPH MINUS A 1-FACTOR

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Abstract. In this paper, we prove that cyclic hamiltonian cycle systems of the complete graph minus a 1-factor, $K_n - I$, exist if and only if $n \equiv 2, 4 (\mod 8)$ and $n \neq 2p^\alpha$ with p prime and $\alpha \geq 1$.

1. Introduction

Throughout this paper, K_n will denote the complete graph on n vertices, $K_n - I$ will denote the complete graph on n vertices with a 1-factor I removed (a 1-*factor* is a 1-regular spanning subgraph), and C_m will denote the m-cycle $(v_1, v_2, \ldots, v_m)$. An *m-cycle system* of a graph G is a set $\mathcal{C}$ of m-cycles in G whose edges partition the edge set of G. An m-cycle system is called *hamiltonian* if $m = |V(G)|$.

Several obvious necessary conditions for an m-cycle system $\mathcal{C}$ of a graph G to exist are immediate: $m \leq |V(G)|$, the degrees of the vertices of G must be even, and m must divide the number of edges in G. A survey on cycle systems is given in [11] and necessary and sufficient conditions for the existence of an m-cycle system of K_n and $K_n - I$ were given in [1, 14] where it was shown that a m-cycle system of K_n or $K_n - I$ exists if and only if $n \geq m$, every vertex of K_n or $K_n - I$ has even degree, and m divides the number of edges in K_n or $K_n - I$, respectively.

Throughout this paper, ρ will denote the permutation $(0\ 1\ \ldots\ n-1)$, so $\langle \rho \rangle = \mathbb{Z}_n$. An m-cycle system $\mathcal{C}$ of a graph G with vertex set $\mathbb{Z}_n$ is *cyclic* if, for every m-cycle $C = (v_1, v_2, \ldots, v_m)$ in $\mathcal{C}$, the m-cycle $\rho(C) = (\rho(v_1), \rho(v_2), \ldots, \rho(v_m))$ is also in $\mathcal{C}$. An n-cycle system $\mathcal{C}$ of a graph G with vertex set $\mathbb{Z}_n$ is called a *cyclic hamiltonian cycle system.* Finding necessary and sufficient conditions for cyclic m-cycle systems of K_n is an interesting problem and has attracted much attention (see, for example, [2, 3, 4, 5, 7, 8, 9, 12]). The obvious necessary conditions for a cyclic m-cycle system of K_n are the same as for an m-cycle system of K_n; that is, $n \geq m \geq 3$, n is odd (so that the degree of every vertex is even), and m must divide the number of edges in K_n. However, these conditions are no longer necessarily sufficient. For example, it is not difficult to see that there is no cyclic decomposition of K_{15} into 15-cycles. Also, if p is an odd prime and $\alpha \geq 2$, then K_{p^α} cannot be decomposed cyclically into p^α-cycles [5].

The existence question for cyclic m-cycle systems of K_n has been completely settled in a few small cases, namely $m = 3$ [10], 5 and 7 [12]. For even m and $n \equiv 1 (\mod 2m)$, cyclic m-cycle systems of K_n are constructed for $m \equiv 0 (\mod 4)$ in [9] and for $m \equiv 2 (\mod 4)$ in

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[12]. Both of these cases are handled simultaneously in [7]. For odd m and $n \equiv 1 (\bmod 2m)$, cyclic m-cycle systems of K_n are found using different methods in [2, 4, 8]. In [3], as a consequence of a more general result, cyclic m-cycle systems of K_n for all positive integers m and $n \equiv 1 (\bmod 2m)$ with $n \geq m \geq 3$ are given using similar methods. Recently, it has been shown [5] that a cyclic hamiltonian cycle system of K_n exists if and only if $n \neq 15$ and $n \notin \{p^\alpha \mid p \text{ is an odd prime and } \alpha \geq 2\}$. Thus, as a consequence of a result in [4], cyclic m-cycle systems of K_{2mk+m} exist for all $m \neq 15$ and $m \notin \{p^\alpha \mid p \text{ is an odd prime and } \alpha \geq 2\}$. In [15], the last remaining cases for cyclic m-cycle systems of K_{2mk+m} are settled, i.e., it is shown that, for $k \geq 1$, cyclic m-cycle systems of K_{2km+m} exist if $m = 15$ or $m \in \{p^\alpha \mid p \text{ is an odd prime and } \alpha \geq 2\}$.

These questions can be extended to the case when n is even by considering the graph $K_n - I$. In [3], it is shown that for all integers $m \geq 3$ and $k \geq 1$, there exists a cyclic m-cycle system of $K_{2mk+2} - I$ if and only if $mk \equiv 0, 3 (\bmod 4)$. In this paper, we are interested in cyclic hamiltonian cycle systems of $K_n - I$ where n is necessarily even. The main result of this paper is the following.

Theorem 1.1. *For an even integer $n \geq 4$, there exists a cyclic hamiltonian cycle system of $K_n - I$ if and only if $n \equiv 2, 4 (\bmod 8)$ and $n \neq 2p^\alpha$ where p is prime and $\alpha \geq 1$.*

Our methods involve circulant graphs and difference constructions. In Section 2, we give some basic definitions and lemmas while the proof of Theorem 1.1 is given in Section 3. In Lemma 3.1, we show that if there is a cyclic hamiltonian cycle system of $K_n - I$, then $n \equiv 2, 4 (\bmod 8)$ and $n \neq 2p^\alpha$ where p is prime and $\alpha \geq 1$. Lemmas 3.2 and 3.3 handle each of these congruence classes modulo 8. Our main theorem then follows.

2. Preliminaries

The proof of Theorem 1.1 uses circulant graphs, which we now define. Let S be a subset of $\mathbb{Z}_n$ satisfying

(1) $0 \notin S$, and
(2) $S = -S$; that is, $s \in S$ implies that $-s \in S$.

The *circulant graph* $X(n; S)$ is defined to be that graph whose vertices are the elements of $\mathbb{Z}_n$, with an edge between vertices g and h if and only if $h = g + s$ for some $s \in S$. We call S the *connection set*, and we will often write $-s$ for $n - s$ when n is understood. Notice that the edge from g to $g+s$ in this graph is generated by both s and $-s$, since $g = (g+s) + (-s)$ and $-s \in S$. Therefore, whenever $S = S' \cup -S'$, where $S' \cap -S' = \{s \in S \mid s = -s\}$, every edge of $X(n; S)$ comes from a unique element of the set S'. Hence we make the following definition. In a circulant graph $X(n; S)$, a set S' with the property that $S = S' \cup -S'$ and $S' \cap -S' = \{s \in S \mid s = -s\}$ is called a *set of edge lengths* for $X(n; S)$.

Notice that in order for a graph G to admit a cyclic m-cycle decomposition, G must be a circulant graph, so circulant graphs provide a natural setting in which to construct cyclic m-cycle decompositions.

The graph K_n is a circulant graph, since $K_n = X(n; \{1, 2, \ldots, n-1\})$. For n even, $K_n - I$ is also a circulant graph, since $K_n - I = X(n; \{1, 2, \ldots, n-1\} \setminus \{n/2\})$ (so the edges of the 1-factor I are of the form $\{i, i + n/2\}$ for $i = 0, 1, \ldots, (n-2)/2$). In fact, if

$n = a'b$ and $\gcd(a', b) = 1$, then we can view $\mathbb{Z}_n$ as $\mathbb{Z}_{a'} \times \mathbb{Z}_b$, using the group isomorphism $\phi : \mathbb{Z}_n \to \mathbb{Z}_{a'} \times \mathbb{Z}_b$ defined by $\phi(k) = (k(\operatorname{mod} a'), k(\operatorname{mod} b))$. We can therefore relabel both the vertices and the edge lengths of the circulant graphs, using ordered pairs from $\mathbb{Z}_{a'} \times \mathbb{Z}_b$, rather than elements of $\mathbb{Z}_n$, by identifying elements of $\mathbb{Z}_n$ with their images under ϕ. This will prove a very useful tool in our results. Throughout Section 3, as n is even, we will use the isomorphism ϕ with $a' = 2a$ for some a, and b odd.

Let H be a subgraph of a circulant graph $X(n; S)$. For a fixed set of edge lengths S', the notation $\ell(H)$ will denote the set of edge lengths belonging to H, that is,

$$\ell(H) = \{s \in S' \mid \{g, g + s\} \in E(H) \text{ for some } g \in \mathbb{Z}_n\}.$$

Many properties of $\ell(H)$ are independent of the choice of S'; in particular, neither of the two lemmas in this section depends on the choice of S'.

Let C be an m-cycle in $X(n; S)$ and recall that the permutation ρ, which generates $\mathbb{Z}_n$, has the property that $\rho(C) \in \mathcal{C}$ whenever $C \in \mathcal{C}$. We can therefore consider the action of $\mathbb{Z}_n$ as a permutation group acting on the elements of $\mathcal{C}$. Viewing matters this way, the *length of the orbit of* C (under the action of $\mathbb{Z}_n$) can be defined as the least positive integer k such that $\rho^k(C) = C$. Observe that such a k exists since ρ has finite order; furthermore, the well-known orbit-stabilizer theorem (see, for example [6, Theorem 1.4A(iii)]) tells us that k divides n. Thus, if G is a graph with a cyclic m-cycle system $\mathcal{C}$ with $C \in \mathcal{C}$ in an orbit of length k, then it must be that k divides $n = |V(G)|$ and that $\rho(C), \rho^2(C), \ldots, \rho^{k-1}(C)$ are distinct m-cycles in $\mathcal{C}$, where $\rho = (0\ 1\ \cdots\ n-1)$.

The next lemma determines $|\ell(C)|$, where C is a cycle in a cyclic m-cycle system of a graph G, and the number of edges of each edge length in $\ell(C)$.

Lemma 2.1. *Let $\mathcal{C}$ be a cyclic m-cycle system of a graph G of order n. If $C \in \mathcal{C}$ is in an orbit of length k, then $|\ell(C)| = mk/n$. Furthermore, if $\ell \in \ell(C)$, then C has n/k edges of length ℓ.*

Proof. Let $\mathcal{C}$ be a cyclic m-cycle system of a graph G of order n. Let $C \in \mathcal{C}$ in an orbit of length k and let $\ell \in \ell(C)$. Thus, $\{i, i + \ell\} \in E(C)$ for some $i \in \mathbb{Z}_n$. Without loss of generality, we may assume that $i = 0$. Since $\rho^k(C) = C$, it follows that $\{k, k + \ell\} \in E(C)$. Likewise, $\{k, k + \ell\} \in E(C)$ implies $\{2k, 2k + \ell\} \in E(C)$. In fact, it must be the case that $\{0, \ell\}, \{k, k + \ell\}, \{2k, 2k + \ell\}, \ldots, \{n - k, n - k + \ell\}$ are distinct edges of C. Thus, C has at least n/k edges of length ℓ.

If $\{j, j + \ell\} \in E(C)$ with $k \nmid j$, then by letting an appropriate power of ρ^k act on this edge, we may assume without loss of generality that $0 < j < k$. But $\rho^j(C) \in \mathcal{C}$, so since $\rho^j(C) \cap C \neq \emptyset$, we must have $\rho^j(C) = C$, contradicting the fact that the length of the orbit of C is k. So C must have exactly n/k edges of length ℓ.

As the choice of $\ell \in \ell(C)$ was arbitrary, it follows that C has n/k edges of length t for all $t \in \ell(C)$. Since C has m edges, we have $(n/k)|\ell(C)| = m$, or $|\ell(C)| = mk/n$. □

In the case that $m = n$, Lemma 2.1 implies that a cycle in an orbit of length k has exactly k distinct edge lengths. More generally, Lemma 2.1 also implies that n/k must divide m; therefore, we have that $(n/k) \mid \gcd(m, n)$.

The next lemma determines the relationship between the length of the orbit of a cycle and the edge lengths appearing on that cycle.

Lemma 2.2. *Let C be an n-cycle in a graph G with $V(G) = \mathbb{Z}_n$ in an orbit of length $k > 1$. Then for each $\ell \in \ell(C)$, we have that $k \nmid \ell$.*

Proof. Let C be an n-cycle in a graph G with $V(G) = \mathbb{Z}_n$ in an orbit of length $k > 1$. Suppose to the contrary there exists $\ell \in \ell(C)$ such that $k \mid \ell$. Then $\rho^\ell(C) = C$. Without loss of generality, we may assume that $\{0, \ell\} \in E(C)$. Thus, $\{\ell, 2\ell\}, \{2\ell, 3\ell\}, \ldots$ are all edges of C. If $d = \gcd(\ell, n) > 1$, then C is a cycle of length $n/d < n$, a contradiction; otherwise C is an n-cycle but $\rho(C) = C$, contradicting the fact that C is in an orbit of length k. Therefore, $k \nmid \ell$. □

Let X be a set of m-cycles in a graph G with vertex set $\mathbb{Z}_n$ such that $\mathcal{C} = \{\rho^i(C) \mid C \in X, i = 0, 1, \ldots, n-1\}$ is an m-cycle system of G. Then X is called a *starter set* for $\mathcal{C}$ and the m-cycles in X are called *starter cycles*. Clearly, every cyclic m-cycle system $\mathcal{C}$ of a graph G has a starter set X as we may always let $X = \mathcal{C}$. A starter set X is called a *minimum starter set* if $C \in X$ implies $\rho^i(C) \notin X$ for $1 \le i \le n-1$.

Let $\mathcal{C}$ be a cyclic m-cycle system of a graph G with $V(G) = \mathbb{Z}_n$. To find a minimum starter set X for $\mathcal{C}$, we start by adding C_1 to X if the length of the orbit of C_1 is maximum among the cycles in $\mathcal{C}$. Next, we add C_2 to X if the length of the orbit of C_2 is maximum among the cycles in $\mathcal{C} \setminus \{\rho^i(C_1) \mid 0 \le i \le n-1\}$. Continuing in this manner, we add C_3 to X if the length of the orbit of C_3 is maximum among the cycles in $\mathcal{C} \setminus \{\rho^i(C_1), \rho^i(C_2) \mid 0 \le i \le n-1\}$. We continue in this manner until $\{\rho^i(C) \mid C \in X, 0 \le i \le n-1\} = \mathcal{C}$. Therefore, every cyclic m-cycle system has a minimum starter set. Observe that if X is a minimum starter set for a cyclic m-cycle system $\mathcal{C}$ of the graph $X(n; S)$ and S' is a set of edge lengths, then it must be that the collection of sets $\{\ell(C) \mid C \in X\}$ forms a partition of S'.

3. Proof of the Main Theorem

In this section, we will prove Theorem 1.1. We begin by determining the admissible values of n in Lemma 3.1. Next, for those admissible values of n, we construct cyclic hamiltonian cycles systems of $K_n - I$ in Lemmas 3.2 and 3.3. The strategy we will adopt is as follows. For n even, we will choose integers a and b so that $n = 2ab$ with b odd and $\gcd(a, b) = 1$. We will then view $K_n - I$ as a circulant graph labelled by the elements of $\mathbb{Z}_{2a} \times \mathbb{Z}_b$. Let

$$S' = \{(0, j), (a, j) \mid 1 \le j \le (b-1)/2\} \cup \{(i, k) \mid 1 \le i \le a-1, 0 \le k \le b-1\},$$

and observe that $|S'| = (b-1) + (a-1)b = ab - 1 = (n-2)/2$. Now $S' \cap -S' = \emptyset$, so that $X(n; \phi^{-1}(S' \cup -S'))$ is an $(n-2)$-regular graph so indeed $X(n; \phi^{-1}(S' \cup -S')) = K_n - I$, and $\phi^{-1}(S')$ is a set of edge lengths of $K_n - I$, which becomes the set S' under relabelling.

Let $\hat{\rho} = \phi\rho\phi^{-1}$ and note that

$$\hat{\rho} = ((0,0)\ (1,1)\ (2,2)\ \cdots\ (2a-1, b-1))$$

generates $\mathbb{Z}_{2a} \times \mathbb{Z}_b$, that is, $\langle \hat{\rho} \rangle = \mathbb{Z}_{2a} \times \mathbb{Z}_b$. Let $\mathcal{C}$ be an m-cycle system of $K_n - I$ where the vertices have been labelled by the elements of $\mathbb{Z}_{2a} \times \mathbb{Z}_b$ such that $C \in \mathcal{C}$ implies $\hat{\rho}(C) \in \mathcal{C}$. Then, clearly $\{\phi^{-1}(C) \mid C \in \mathcal{C}\}$ is a cyclic m-cycle system of $K_n - I$.

Next observe that if $(e, f) \in S'$ has $\gcd(e, 2a) = 1$ and $\gcd(f, b) = 1$, then $X\,(n; \{\pm\phi^{-1}((e, f))\})$, the subgraph consisting of the edges of length $\pm\phi^{-1}((e, f))$, forms an n-cycle C with the property that $\rho(C) = C$. Let

$$T = \{(i, j) \in S' \mid \gcd(i, 2a) > 1 \text{ or } \gcd(j, b) > 1\}.$$

To find a cyclic hamiltonian cycle system of $K_n - I$, it suffices to find a set X of n-cycles such that $\{\ell(C) \mid C \in X\}$ is a partition of T. Then the collection

$$\mathcal{C} = \{\phi^{-1}(C), \rho(\phi^{-1}(C)), \ldots, \rho^{n-1}(\phi^{-1}(C)) \mid C \in X\} \cup \{X\left(n; \{\pm\phi^{-1}((e, f))\}\right) \mid (e, f) \in S' \setminus T\}$$

is a cyclic hamiltonian cycle system of $K_n - I$.

We now show that if $K_n - I$ has a cyclic hamiltonian cycle system for n even, then $n \geq 4$ with $n \equiv 2, 4 \pmod 8$ and $n \neq 2p^\alpha$ where p is prime and $\alpha \geq 1$.

Lemma 3.1. *For an even integer $n \geq 4$, if there exists a cyclic hamiltonian cycle system of $K_n - I$, then $n \equiv 2, 4 \pmod 8$ and $n \neq 2p^\alpha$ where p is prime and $\alpha \geq 1$.*

Proof. Let $n \geq 4$ be an even integer and suppose that $K_n - I$ has a cyclic hamiltonian cycle system $\mathcal{C}$. Let X be a minimum starter set for $\mathcal{C}$ and let $C \in X$ be in an orbit of length k. Let $P : 0, v_1, v_2, \ldots, v_k$ be the subpath of C, starting at vertex 0, of length k. We wish to show that the edge lengths of P are distinct. Suppose, to the contrary, that two edges of P have the same length ℓ. Then, since $P, \rho^k(P), \rho^{2k}(P), \ldots, \rho^{n-k}(P)$ are pairwise edge-disjoint subpaths of C, it follows that C has $2n/k$ edges of length ℓ, contradicting Lemma 2.1. Thus, $\ell(P) = \ell(C)$.

Next, since $\rho^{ik}(P)$ is a subpath of C, an n-cycle, for $1 \leq i \leq n/k - 1$, it follows that $v_k = jk$ for some positive integer j. Thus, P begins at vertex 0 and ends at vertex jk. Also, since C is an n-cycle, we have that $\gcd(j, n/k) = 1$. Suppose first that k is odd. Then n/k is even since n is even. Thus, since $\gcd(j, n/k) = 1$, it follows that j is odd. Hence jk is odd, so that the number of odd edge lengths in $\ell(C)$ is odd. Since $|\ell(C)| = k$ is odd, it follows that $\ell(C)$ has an even number of even edge lengths. Next, if k is even, then jk is even so that $\ell(C)$ has an even number of odd edge lengths and hence $\ell(C)$ must have an even number of even edge lengths.

Thus, if $C \in X$, then $\ell(C)$ has an even number of even edge lengths. Since $\{\ell(C) \mid C \in X\}$ is a partition of $\{1, 2, \ldots, (n-2)/2\}$, it follows that there must be an even number of even integers in the set $\{1, 2, \ldots, (n-2)/2\}$. Since n is even, we have that $n \equiv 2, 4 \pmod 8$.

It remains to show that $n \neq 2p^\alpha$ where p is prime and $\alpha \geq 1$. Suppose, to the contrary, that $n = 2p^\alpha$ for some prime p and $\alpha \geq 1$. Let X be a minimum starter set for $\mathcal{C}$ and choose $C \in X$ with $2p^{\alpha-1} \in \ell(C)$ (replace S' by $-S'$ if necessary to ensure that $2p^{\alpha-1} \in S'$). Suppose that C is in an orbit of length k. Then $k \mid 2p^\alpha$, and since $K_n - I$ has $2p^\alpha(2p^\alpha - 2)/2$ edges and each cycle of $\mathcal{C}$ has $2p^\alpha$ edges, we must have $|\mathcal{C}| = p^\alpha - 1$. It therefore follows that $1 \leq k < 2p^\alpha$. Hence, $k \mid 2p^{\alpha-1}$, and by Lemma 2.2, we must have $k = 1$. But if $k = 1$, then $\ell(C) = \{2p^{\alpha-1}\}$ and since $X(2p^\alpha; \{\pm 2p^{\alpha-1}\})$ consists of $2p^{\alpha-1}$ p-cycles, we have a contradiction. Therefore, $n \neq 2p^\alpha$ where p is prime and $\alpha \geq 1$. □

We will handle each of the cases $n \equiv 2 \pmod 8$ and $n \equiv 4 \pmod 8$ separately. We begin with the case $n \equiv 4 \pmod 8$ as this is the easier of the two cases.

Lemma 3.2. *For $n \equiv 4 \pmod 8$, the graph $K_n - I$ has a cyclic hamiltonian cycle system.*

Proof. Suppose that $n \equiv 4 (\mathrm{mod}\, 8)$, say $n = 8q + 4$ for some nonnegative integer q. Since $K_4 - I$ is a 4-cycle, we may assume that $q \geq 1$. Now, $\mathbb{Z}_n \cong \mathbb{Z}_4 \times \mathbb{Z}_{2q+1}$ and thus we will use ϕ to relabel the vertices of $K_n - I = X(n; \{1, \ldots, n-1\} \setminus \{n/2\})$ with the elements of $\mathbb{Z}_4 \times \mathbb{Z}_{2q+1}$. The set

$$S' = \{(0, i), (2, i) \mid 1 \leq i \leq q\} \cup \{(1, j) \mid 0 \leq j \leq 2q\}$$

has the property that $S' \cap -S' = \emptyset$ and $\phi^{-1}(S' \cup -S') = \{1, 2, \ldots, n-1\} \setminus \{n/2\}$. Thus we can think of the elements of S' as the edge lengths of the relabelled graph. If q is even, say $q = 2j$ for some positive integer j, define the walk P by

$$\begin{aligned} P \ : \ & (0,0), (0,1), (0,-1), (0,2), (0,-2), \ldots, (0,j), (0,-j), \\ & (2, j+1), (0, -(j+1)), (2, j+2), (0, -(j+2)), \ldots, (2, q), (0, -q), (1, 0). \end{aligned}$$

If q is odd, say $q = 2j + 1$ for some positive integer j, define the walk P by

$$\begin{aligned} P \ : \ & (0,0), (0,1), (0,-1), (0,2), (0,-2), \ldots, (0,j), (0,-j), (0, j+1), \\ & (2, -(j+1)), (0, j+2), (2, -(j+2)), \ldots, (0, q), (2, -q), (3, 0). \end{aligned}$$

In either case, note that the vertices of P, except for the first and the last, are distinct modulo $2q + 1$ while the first and the last vertices are distinct modulo 4. Therefore, P is a path. Next, the edge lengths of P, in the order they are encountered, are $(0, 1), (0, 2), \ldots, (0, q), (2, q), (2, q-1), \ldots, (2, 1), (1, q)$. Let

$$C = P \cup \hat{\rho}^{2q+1}(P) \cup \hat{\rho}^{4q+2}(P) \cup \hat{\rho}^{6q+3}(P).$$

Then, clearly C is an n-cycle in an orbit of length $2q + 1$ and

$$\ell(C) = \{(0, 1), (0, 2), \ldots, (0, q), (2, q), (2, q-1), \ldots, (2, 1), (1, q)\}.$$

Now, let $d_0, d_1, \ldots, d_t$ denote the integers with $0 \leq d_j < 2q$ and $\gcd(d_j, 2q+1) > 1$. For $j = 0, 1, \ldots, t$, consider the walk $P_j : (0, 0), (1, d_j), (2, 2q)$. Clearly, P_j is a path and the edge lengths of P_j, in the order they are encountered, are $(1, d_j), (1, 2q - d_j)$. Let

$$C_j = P_j \cup \hat{\rho}^2(P_j) \cup \hat{\rho}^4(P_j) \cup \hat{\rho}^6(P-j) \cup \cdots \hat{\rho}^{8q+2}(P_j).$$

Then C_j is an n-cycle in an orbit of length 2 and

$$\ell(C_j) = \{(1, d_j), (1, 2q - d_j)\}.$$

Since $\gcd(q, 2q + 1) = 1$, we have that $d_j \neq q$ and thus $\ell(C) \cap \ell(C_j) = \emptyset$ for $0 \leq j \leq t$.

Let $T = \{\ell(C), \ell(C_0), \ldots, \ell(C_t)\}$. and let $(e, f) \in S' \backslash T$. Then $e = 1$ and $\gcd(f, 2q+1) = 1$. Thus,

$$X = \{\phi^{-1}(C), \phi^{-1}(C_0), \ldots, \phi^{-1}(C_t)\} \cup \{X(n; \{\pm\phi^{-1}((e, f))\}) \mid (e, f) \in S' \setminus T\}$$

is a minimum starter set for a cyclic hamiltonian cycle system of $K_n - I$. □

Before continuing, let Φ denote the Euler-phi function, that is, for a positive integer a, $\Phi(a)$ denotes the number of integers n with $1 \leq n \leq a$ and $\gcd(n, a) = 1$. For a positive integer a, $\Phi(a)$ is easily computed from the prime factorization of a. Let $a = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ where $p_1, p_2, \ldots, p_t$ are distinct primes and $k_1, k_2, \ldots, k_t$ are positive integers. Then

$$\Phi(a) = \prod_{i=1}^{t} p_i^{k_i - 1}(p_i - 1).$$

We now handle the case when $n \equiv 2 \pmod 8$.

Lemma 3.3. *For $n \equiv 2 \pmod 8$ with $n \geq 4$ and $n \neq 2p^\alpha$ where p is prime and $\alpha \geq 1$, the graph $K_n - I$ has a cyclic hamiltonian cycle system.*

Proof. Suppose that $n \equiv 2 \pmod 8$ with $n \neq 2p^\alpha$ where p is prime and $\alpha \geq 1$, say $n = 8q+2$ for some positive integer q. Let $4q+1 = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$ where $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$ are all distinct primes with $r, s \geq 0$, $p_1 < p_2 < \ldots < p_r$, $p_i \equiv 3 \pmod 4$, $k_i \geq 1$ for $1 \leq i \leq r$, $q_i \equiv 1 \pmod 4$, and $j_i \geq 1$ for $1 \leq i \leq s$. Since $n \equiv 2 \pmod 8$, it follows that $\sum k_i$ is even.

Case 1. *Suppose that $s \geq 1$, or some k_i is even for $1 \leq i \leq r$, or $r > 2$.* Let

$$a = \begin{cases} q_1^{j_1} & \text{if } s \geq 1, \\ p_i^{k_i} & \text{if } s = 0 \text{ and } k_i \text{ is even for some } 1 \leq i \leq r, \text{ or} \\ p_2^{k_2} p_3^{k_3} & \text{if } s = 0,\ k_i \text{ is odd for } 1 \leq i \leq r, \text{ and } r > 2. \end{cases}$$

Note that for each choice of a, we have that $a \equiv 1 \pmod 4$. Let $b = (4q+1)/a$ and observe that $\gcd(a, b) = 1$. Next, we will use ϕ to relabel the vertices of $K_n - I = X(n; \{1, \ldots, n-1\} \setminus \{n/2\})$ with the elements of $\mathbb{Z}_{2a} \times \mathbb{Z}_b$. The set

$$S' = \{(0, j), (a, j) \mid 1 \leq j \leq (b-1)/2\} \cup \{(i, j) \mid 1 \leq i \leq a-1, 0 \leq j \leq b-1\}$$

has the property that $S' \cap -S' = \emptyset$ and $\phi^{-1}(S' \cup -S') = \{1, 2, \ldots, n-1\} \setminus \{n/2\}$, so we can think of the elements of S' as the edge lengths of the relabelled graph.

Let $d_1, d_2, \ldots, d_t$ denote the integers with $1 \leq d_j < a$ and $\gcd(d_j, 2a) > 1$ and let $e_1, e_2, \ldots, e_{a-1-t}$ denote the integers in the set $\{1, 2, \ldots, a-1\} \setminus \{d_1, d_2, \ldots, d_t\}$ so that $(e_i, 2a) = 1$ for $1 \leq i \leq a-1-t$. We will need to show that $2(a-1-t) \geq t+1$.

First, $\Phi(2a)$ is the number of integers n with $1 \leq n \leq 2a$ and $\gcd(n, 2a) = 1$. Thus, $2a - \Phi(2a)$ is the number of integers n with $1 \leq n \leq 2a$ and $\gcd(n, 2a) > 1$ so that $(2a - \Phi(2a))/2$ is the number of integers n with $1 \leq n \leq a$ and $\gcd(n, 2a) > 1$. Hence $t = (2a - \Phi(2a))/2 - 1$, since each $d_i < a$. Substituting $t = (2a - \Phi(2a))/2 - 1$ into $2(a-1-t) \geq t+1$, we obtain the inequality $\Phi(2a) \geq 2a/3$, which needs to be verified for each choice of a above. Suppose first that $a = q_1^{j_1}$. Then, since $q_1 \geq 5 > 3$ and $\Phi(2a) = q_1^{j_1 - 1}(q_1 - 1)$, it easily follows that $\Phi(2a) \geq 2a/3$. Similarly, if $a = p_i^{k_i}$, then again $\Phi(2a) \geq 2a/3$ since $p_i \geq 3$. Next suppose that $a = p_2^{k_2} p_3^{k_3}$ and observe that since $p_3 > p_2 > p_1$, it follows that $p_2 \geq 7$ and $p_3 \geq 11$. Now $\Phi(2a) \geq 2a/3$ is equivalent to $\Phi(2a)/a \geq 2/3$, and since $\Phi(2a) = p_2^{k_2-1}(p_2 - 1) p_3^{k_3-1}(p_3 - 1)$, it follows that $\Phi(2a)/a = (p_2 - 1)(p_3 - 1)/(p_2 p_3) \geq 60/77 > 2/3$. Hence, $\Phi(2a) \geq 2a/3$ if $a = p_2^{k_2} p_3^{k_3}$.

Let $b = 2m+1$ for some positive integer m. Since $b = (4q+1)/a$ and $a \equiv 1 \pmod 4$, we also have $b \equiv 1 \pmod 4$, so m is even. Say $m = 2j$ for some positive integer j, and define the walk P by

$$\begin{aligned} P \ : \ & (0,0),(0,1),(0,-1),(0,2),(0,-2),\dots,(0,j),(0,-j), \\ & (a,j+1),(0,-(j+1)),(a,j+2),(0,-(j+2)),\dots,(a,m),(0,-m),(e_1,0). \end{aligned}$$

Note that the vertices of P, except for the first and the last, are distinct modulo b, while the first and the last vertices are distinct modulo $2a$. Therefore, P is a path. Next, the edge lengths of P, in the order they are encountered, are $(0,1),(0,2),\dots,(0,m),(a,m)$, $(a,m-1),\dots,(a,1),(e_1,m)$. Let

$$C = P \cup \hat{\rho}^b(P) \cup \hat{\rho}^{2b}(P) \cup \cdots \hat{\rho}^{(2a-1)b}(P).$$

Since the last vertex of P is $(e_1,0)$, and $\gcd(e_1,2a)=1$, we have that C is an n-cycle in an orbit of length b where

$$\ell(C) = \{(0,1),(0,2),\dots,(0,m),(a,m),(a,m-1),\dots,(a,1),(e_1,m)\}.$$

Now, define the walks $P_1,P_2,\dots,P_t$ as follows for $i=1,3,5,\dots$,

$$P_i \ : \ (0,0),(d_i,1),(0,-1),(d_i,2),(0,-2),\dots,(d_i,m),(0,-m),(-e_{(i+1)/2},0),$$

and

$$P_{i+1} \ : \ (0,0),(d_{i+1},1),(0,-1),(d_{i+1},2),(0,-2),\dots,(d_{i+1},m),(0,-m),(e_{(i+1)/2+1},0).$$

For $j=1,2,\dots,t$, the vertices of P_j, except for the first and the last, are distinct modulo b, while the first and the last vertices are distinct modulo $2a$. Therefore, P_j is a path. Next, the edge lengths of P_j, in the order they are encountered, are $(d_j,1),(d_j,2),\dots,(d_j,m),(d_j,m+1),\dots,(d_j,b-1)$, and $(e_{(j+1)/2},m+1)$ if j is odd or $(e_{j/2+1},m)$ if j is even. Let

$$C_j = P_j \cup \hat{\rho}^b(P_j) \cup \hat{\rho}^{2b}(P_j) \cup \cdots \hat{\rho}^{(2a-1)b}(P_j).$$

Since the last vertex $(k,0)$ of P_j, where $k=-e_{(j+1)/2}$ or $k=e_{j/2+1}$ has the property that $\gcd(k,2a)=1$, we have that C_j is an n-cycle in an orbit of length b where

$$\ell(C_j) = \{(d_j,1),(d_j,2),\dots,(d_j,m),(d_j,m+1),\dots,(d_j,b-1),(e_{(j+1)/2},m+1)\}$$

if j is odd, or

$$\ell(C_j) = \{(d_j,1),(d_j,2),\dots,(d_j,m),(d_j,m+1),\dots,(d_j,b-1),(e_{j/2+1},m)\}$$

if j is even.

Define the set $A = \ell(C) \cup \ell(C_1) \cup \ell(C_2) \cup \cdots \cup \ell(C_t)$. Now, A contains $t+1$ elements from the set $\{(e_i,m),(e_i,m+1) \mid 1 \le i \le a-1-t\}$. Next

$$|\{(e_i,m),(e_i,m+1) \mid 1 \le i \le a-1-t\}| = 2(a-1-t).$$

Since we have seen previously that $2(a-1-t) \ge t+1$, it follows that there are enough distinct values of e_i to make edge lengths in A distinct, so $|A| = (t+1)b$.

Let $c_1,c_2,\dots,c_x$ denote the integers with $1 \le c_j < b$ and $\gcd(c_j,b) > 1$ for $1 \le j \le x$. Fix j with $1 \le j \le x$ and for $i=1,2,\dots,a-1-t$, consider the walk $P_{i,j} : (0,0),(e_i,c_j),(2e_i,b-1)$. Clearly, $P_{i,j}$ is a path and the edge lengths of $P_{i,j}$, in the order they are encountered, are $(e_i,c_j),(e_i,b-1-c_j)$. Let

$$C_{i,j} = P_{i,j} \cup \hat{\rho}^2(P_{i,j}) \cup \hat{\rho}^4(P_{i,j}) \cup \hat{\rho}^6(P_{i,j}) \cup \cdots \hat{\rho}^{2ab-2}(P_{i,j}).$$

Since $\gcd(e_i, a) = 1$, it follows that $C_{i,j}$ is an n-cycle in an orbit of length 2 and

$$\ell(C_{i,j}) = \{(e_i, c_j), (e_i, b - 1 - c_j)\}.$$

Define the set

$$B = \bigcup_{\substack{1 \le i \le a-1-t \\ 1 \le j \le x}} \ell(C_{i,j}).$$

We want $A \cap B = \emptyset$. Now, if $A \cap B \neq \emptyset$, then as $\gcd(c_k, b) > 1$ for every k and $b = 2m + 1$, we cannot have $c = m$ or $c = m + 1$, so it must be the case that $b - 1 - c_k = m + 1$ for some k with $1 \le k \le x$. Thus $c_k = (b - 3)/2 = m - 1$. In this case, for $i = 1, 2, \ldots, a - 1 - t$, define $P_{i,k} : (0, 0), (e_i, c_k), (2e_i, m)$ and create $C_{i,k}$ as before. Thus

$$\ell(C_{i,k}) = \{(e_i, c_k), (e_i, 1)\}.$$

Since $\gcd(2e_i, 2a) = 2$, it follows that $C_{i,k}$ will be an n-cycle in an orbit of length 2. Thus $A \cap B = \emptyset$.

Finally, consider the path $P' : (0, 0), (1, 0), (-1, 0), (2, 0), (-2, 0), \ldots, ((a-1)/2, 0), (-(a-1)/2, 0), (a, 1)$ and let

$$C' = P' \cup \hat{\rho}^{a}(P') \cup \hat{\rho}^{2a}(P') \cup \cdots \cup \hat{\rho}^{a(2b-1)}(P').$$

Since $\gcd(1, b) = 1$, we have that C' is an n-cycle in an orbit of length a and

$$\ell(C') = \{(1, 0), (2, 0), \ldots, (a - 1, 0), ((a + 1)/2, b - 1)\}.$$

Since $a \equiv 1 (\operatorname{mod} 4)$ we have that $\gcd((a+1)/2, 2a) = 1$ and therefore $((a+1)/2, b-1) \notin A \cup B$.

Let $T = S' \setminus (A \cup B \cup \ell(C'))$ and let $(e, f) \in T$. Then, it must be that $\gcd(e, 2a) = 1$ and $\gcd(f, b) = 1$. Thus,

$$\begin{aligned} X \;=\; & \{\phi^{-1}(C), \phi^{-1}(C_1), \ldots \phi^{-1}(C_t), \phi^{-1}(C_{1,1}), \phi^{-1}(C_{1,2}), \ldots, \phi^{-1}(C_{1,x}), \phi^{-1}(C_{2,1}), \phi^{-1}(C_{2,2}), \\ & \ldots, \phi^{-1}(C_{2,x}), \ldots, \phi^{-1}(C_{a-1-t,1}), \phi^{-1}(C_{a-1-t,2}), \ldots, \phi^{-1}(C_{a-1-t,x}), \phi^{-1}(C')\} \\ & \bigcup \{X(n; \{\pm\phi^{-1}((e, f))\}) \mid (e, f) \in T\} \end{aligned}$$

is a minimum starter set for for a cyclic hamiltonian cycle system of $K_n - I$.

Case 2. *Suppose that $s = 0$, k_i is odd for $1 \le i \le r$, and $r = 2$.* Thus $n = 2p_1^{k_1} p_2^{k_2}$ where k_1 and k_2 are odd. In this case, we will let $a = p_1^{k_1}$, $b = p_2^{k_2}$ and use ϕ to relabel the vertices of $K_n - I = X(n; \{1, \ldots, n - 1\} \setminus \{n/2\})$ with the elements of $\mathbb{Z}_{2a} \times \mathbb{Z}_b$. The set

$$S' = \{(0, j), (a, j) \mid 1 \le j \le (b - 1)/2\} \cup \{(i, j) \mid 1 \le i \le a - 1, 0 \le j \le b - 1\}$$

has the property that $\phi^{-1}(S')$ is a set of edge lengths of $K_n - I$, so we can think of the elements of S' as the edge lengths of the relabelled graph.

Let $d_1, d_2, \ldots, d_t$ denote the integers with $1 \le d_j < a$ and $\gcd(d_j, 2a) > 1$ and let $e_1, e_2, \ldots, e_{a-1-t}$ denote the integers in the set $\{1, 2, \ldots, a - 1\} \setminus \{d_1, d_2, \ldots, d_t\}$ so that $\gcd(e_i, 2a) = 1$ for $1 \le i \le a - 1 - t$. In this case, note that as $p_1 \equiv 3 (\operatorname{mod} 4)$ and k_1 is odd, $\gcd((a + 1)/2, 2a) = 2$ so that $(a + 1)/2 \in \{d_1, d_2, \ldots, d_t\}$. Without loss of generality, let $d_1 = (a + 1)/2$ and $e_1 = 1$.

Since k_2 is odd and $p_2 \equiv 3 (\bmod 4)$, it follows that $b = p_2^{k_2} = 4j+3$ for some positive integer j. Define the walk P by

$$\begin{aligned} P \quad : \quad & (0,0),(0,1),(0,-1),(0,2),(0,-2),\dots,(0,j),(0,-j),(0,j+1),(a,-(j+1)),\\ & (0,j+2),(a,-(j+2)),\dots,(0,2j+1),(a,-(2j+1)),((3a+1)/2,0). \end{aligned}$$

Note that the vertices of P, except for the first and the last, are distinct modulo b, while the first and the last vertices are distinct modulo $2a$. Therefore, P is a path. Next, the edge lengths of P, in the order they are encountered, are $(0,1),(0,2),\dots,(0,2j+1),(a,2j+1)$, $(a,2j),\dots,(a,1),((a+1)/2,2j+1)$. Let

$$C = P \cup \hat{\rho}^b(P) \cup \hat{\rho}^{2b}(P) \cup \cdots \hat{\rho}^{(2a-1)b}(P).$$

Since the last vertex $((3a+1)/2,0)$ of P has the property that $\gcd((3a+1)/2,2a) = 1$, we have that C is an n-cycle in an orbit of length b where

$$\ell(C) = \{(0,1),(0,2),\dots,(0,2j+1),(a,2j+1),(a,2j),\dots,(a,1),((a+1)/2,2j+1)\}.$$

Define the walk P_1 by

$$\begin{aligned} P_1 \quad : \quad & (0,0),((a+1)/2,1),(0,-1),((a+1)/2,2),(0,-2),\dots,((a+1)/2,j),(0,-j),\\ & (1,j+1),(1-(a+1)/2,-(j+1)),(1,j+2),(1-(a+1)/2,-(j+2)),\\ & \dots,(1,2j+1),(0,-(2j+1)),(e_{a-1-t},0). \end{aligned}$$

Now, for $i = 2,3,\dots,t-1$, define the walk P_i by

$$P_i \quad : \quad (0,0),(d_i,1),(0,-1),(d_i,2),(0,-2),\dots,(d_i,2j+1),(0,-(2j+1)),(e_{(i+1)/2},0),$$

if i is odd, or

$$P_i \quad : \quad (0,0),(d_i,1),(0,-1),(d_i,2),(0,-2),\dots,(d_i,2j+1),(0,-(2j+1)),(-e_{i/2},0)$$

if i is even. Define the walk P_t by

$$P_t \quad : \quad (0,0),(d_t,1),(0,-1),(d_t,2),(0,-2),\dots,(d_t,2j+1),(0,-(2j+1)),(-e_{a-1-t},0).$$

For $i = 1,2,\dots,t$, the vertices of P_i, except for the first and the last, are distinct modulo b, while the first and the last vertices are distinct modulo $2a$. Therefore, P_i is a path. Next, the edge lengths of P_i for $i \neq 1$, in the order they are encountered, are $(d_i,1),(d_i,2),\dots,(d_i,2j+1),(d_i,2j+2),\dots,(d_i,b-1)$, and $(e_{(i+1)/2},2j+1)$ if $1 < i < t$ is odd, $(e_{i/2},2j+2)$ if $i < t$ is even, or $(e_{a-1-t},2j+2)$ if $i = t$. For $i = 1$, the edge lengths of P_i, in the order they are encountered, are $(d_i,1),(d_i,2),\dots,(d_i,2j),(1,2j+1),(d_i,2j+2),\dots,(d_i,b-1)$, and $(e_{a-1-t},2j+1)$. Let

$$C_i = P_i \cup \hat{\rho}^b(P_i) \cup \hat{\rho}^{2b}(P_i) \cup \cdots \hat{\rho}^{(2a-1)b}(P_i).$$

Since the last vertex $(\ell,0)$ of P_i has the property that $\gcd(\ell,2a) = 1$, we have that C_i is an n-cycle in an orbit of length b where

$$\ell(C_i) = \{(d_i,1),(d_i,2),\dots,(d_i,2j+1),(d_i,2j+2),\dots,(d_i,b-1),(e_{(i+1)/2},2j+1)\}$$

if i is odd and $1 < i < t$,

$$\ell(C_i) = \{(d_i,1),(d_i,2),\dots,(d_i,2j+1),(d_i,2j+2),\dots,(d_i,b-1),(e_{i/2},2j+2)\}$$

if i is even and $1 < i < t$,

$$\ell(C_i) = \{(d_i, 1), (d_i, 2), \ldots, (d_i, 2j), (1, 2j+1), (d_i, 2j+2), \ldots, (1, b-1), (e_{a-1-t}, 2j+1)\}$$

if $i = 1$, or

$$\ell(C_i) = \{(d_i, 1), (d_i, 2), \ldots, (d_i, 2j+1), (d_i, 2j+2), \ldots, (d_i, b-1), (e_{a-1-t}, 2j+2)\}$$

if $i = t$.

Define the set $A = \ell(C) \cup \ell(C_1) \cup \ell(C_2) \cup \cdots \cup \ell(C_t)$. Now, A contains $t+1$ elements from the set $\{(e_i, 2j+1), (e_i, 2j+2) \mid 1 \leq i \leq a-1-t\}$. Again

$$|\{(e_i, 2j+1), (e_i, 2j+2) \mid 1 \leq i \leq a-1-t\}| = 2(a-1-t).$$

As in Case 1, $t = (2a - \Phi(2a))/2$ where in this case $a = p_1^{k_1}$, and we need $2(a-1-t) \geq t+1$. Since $p_1 \geq 3$, the inequality follows. So there are enough distinct values of e_i to make edge lengths in A distinct and therefore $|A| = (t+1)b$.

Let $c_1, c_2, \ldots, c_x$ denote the integers with $1 \leq c_j < b$ and $\gcd(c_j, b) > 1$ for $1 \leq j \leq x$. Fix j with $1 \leq j \leq x$ and for $i = 1, 2, \ldots, a-1-t$, consider the walk $P_{i,j} : (0,0), (e_i, c_j), (2e_i, b-1)$. Clearly, $P_{i,j}$ is a path and the edge lengths of $P_{i,j}$, in the order they are encountered, are $(e_i, c_j), (e_i, b-1-c_j)$. Let

$$C_{i,j} = P_{i,j} \cup \hat{\rho}^2(P_{i,j}) \cup \hat{\rho}^4(P_{i,j}) \cup \hat{\rho}^6(P_{i,j}) \cup \cdots \hat{\rho}^{2ab-2}(P_{i,j}).$$

Since $\gcd(e_i, a) = 1$, it follows that $C_{i,j}$ is an n-cycle in an orbit of length 2 and

$$\ell(C_{i,j}) = \{(e_i, c_j), (e_i, b-1-c_j)\}.$$

Define the set

$$B = \bigcup_{\substack{1 \leq i \leq a-1-t \\ 1 \leq j \leq x}} \ell(C_{i,j}).$$

We want $A \cap B = \emptyset$. Now, if $A \cap B \neq \emptyset$, then as $\gcd(c_k, b) > 1$ for every k and $b = 4j+3$, we cannot have $c = 2j+1$ or $c = 2j+2$, so it must be the case that $b-1-c_k = 2j+2$ for some k with $1 \leq k \leq x$. Then $c_k = (b-3)/2$. This implies that $3 \mid p_2^{k_2}$ since $\gcd(c_k, p_2^{k_2}) > 1$. This is impossible since $p_2 \geq 7$.

Finally, consider the path $P' : (0,0), (1,0), (-1,0), (2,0), (-2,0), \ldots, ((a-1)/2, 0), (-(a-1)/2, 0), (a, 1)$ and let

$$C' = P' \cup \hat{\rho}^a(P') \cup \hat{\rho}^{2a}(P') \cup \cdots \cup \hat{\rho}^{a(2b-1)}(P').$$

Since $\gcd(1, b) = 1$, it follows that C' is an n-cycle in an orbit of length a and

$$\ell(C') = \{(1,0), (2,0), \ldots, (a-1, 0), ((a+1)/2, b-1)\}.$$

Let $T = S' \setminus (A \cup B \cup \ell(C'))$ and let $(e, f) \in T$. Then, it must be that $\gcd(e, 2a) = 1$ and $\gcd(f, b) = 1$. Thus,

$$\begin{aligned} X \;=\; & \{\phi^{-1}(C), \phi^{-1}(C_1), \ldots \phi^{-1}(C_t), \phi^{-1}(C_{1,1}), \phi^{-1}(C_{1,2}), \ldots, \phi^{-1}(C_{1,x}), \phi^{-1}(C_{2,1}), \phi^{-1}(C_{2,2}), \\ & \ldots, \phi^{-1}(C_{2,x}), \ldots, \phi^{-1}(C_{a-1-t,1}), \phi^{-1}(C_{a-1-t,2}), \ldots, \phi^{-1}(C_{a-1-t,x}), \phi^{-1}(C')\} \\ & \bigcup \{X(n; \{\pm\phi^{-1}((e,f))\}) \mid (e,f) \in T\} \end{aligned}$$

is a minimum starter set for for a cyclic hamiltonian cycle system of $K_n - I$.

□

Theorem 1.1 now follows from Lemmas 3.1, 3.2, and 3.3.

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