Isomorphic Cayley Graphs on different Groups

Joy Morris
Department of Mathematics and Statistics,
Simon Fraser University,
Burnaby, BC. V5A 1S6. CANADA.
morris@cs.sfu.ca

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Abstract:
The issue of when two Cayley digraphs on different abelian groups of prime power order can be isomorphic is examined. This had previously been determined by Anne Joseph for squares of primes; her results are extended.

1 Preliminaries

We begin with some essential definitions. For many of these results, the lemmata and proofs used are direct extensions of those in Joseph's paper [1]. Although we are dealing with abelian groups, multiplicative notation will be used.

Definition. Let S be a subset of a group G. The *Cayley digraph* $X = X(G; S)$ is the directed graph given as follows. The vertices of X are the group elements of G. There is an arc between two vertices g and h if and only if $g^{-1}h \in S$. In other words, for every vertex $g \in G$ and element $s \in S$, there is an arc from g to gs.

Notice that if the identity element 1 of G is in S, then there is a loop at every vertex, while if $1 \notin S$, the digraph has no loops. For convenience, we will assume the latter case holds; it makes no difference to the results. Also notice that since S is a set, it contains no multiple entries and hence there are no multiple arcs.

Definition. The *wreath product* of two digraphs X and Y, written $X \wr Y$, is given as follows. The vertices of the new graph are all pairs (x, y) where x is a vertex of X and y is a vertex of Y. The arcs of $X \wr Y$ are given by the pairs $\{[(x_1, y_1), (x_1, y_2)] : [y_1, y_2]$ is an arc of $Y\}$ together with $\{[(x_1, y_1), (x_2, y_2)] : [x_1, x_2]$ is an arc of $X\}$. In other words, there are $|V(X)|$ copies of the digraph Y, and arcs exist from one copy of Y to another if and only if there is an arc in the same direction between the corresponding vertices of X. If any arcs exist from one copy of Y to another, then all arcs exist in that direction between those copies of Y.

We denote the direct product of n copies of the cyclic group of integers modulo p by $(\mathbf{Z}_p)^n$.

We are now ready to give the main result.

Theorem 1.1 *Let $X = X(G; S)$ be a Cayley digraph on an abelian group G of order p^n, where p is prime and $n \geq 2$. Then the following are equivalent:*

1. *The digraph X is isomorphic to a Cayley digraph on both $\mathbf{Z}_{p^n}$ and $(\mathbf{Z}_p)^n$.*
2. *There exist successive subgroups $H_1 \subset \ldots \subset H_{n-1}$ in G such that $|H_i| = p^i$ and for all $s \in S \setminus H_i, sH_i \subseteq S, i = 1, \ldots, n-1$. (That is, $S \setminus H_i$ is a union of cosets of H_i.)*
3. *There exist Cayley digraphs $U_1, \ldots, U_n$ on $\mathbf{Z}_p$ such that X is isomorphic to $U_1 \wr \ldots \wr U_n$.*

   These in turn imply:

4. *X is isomorphic to Cayley digraphs on every direct product of groups $\mathbf{Z}_{p^{k_1}} \times \mathbf{Z}_{p^{k_2}} \times \ldots \times \mathbf{Z}_{p^{k_m}}$ where $2 \leq m \leq n-1$ and $k_1+k_2+\ldots+k_m = n$. That is, on every abelian group of order p^n.*

2 Method of proof

Throughout the proof of the main result, we will generally be using induction. The base case is $n = 2$, and is provided in Anne Joseph's paper [1]. Again, we begin this section with some necessary preliminaries.

Definition. The action of a group G on a set V is *regular* if it is transitive and $|V| = |G|$. Equivalently, if for every $x, y \in V$, there is a unique $g \in G$ such that $g(a) = b$.

Theorem 2.1 (See [1] or [2]) *Let X be a digraph and G be a group. The automorphism group Aut(X) has a subgroup isomorphic to G which acts regularly on $V(X)$ if and only if X is isomorphic to a Cayley digraph $X(G; S)$ for some subset S of G.*

Thus, since X is a Cayley graph on both $\mathbf{Z}_{p^n}$ and $(\mathbf{Z}_p)^n$, the group $\mathrm{Aut}(X)$ contains a regular subgroup which is isomorphic to $\mathbf{Z}_{p^n}$. This is certainly contained in a Sylow p-subgroup.

Definition. In a set X under the action of a group G, a proper subset B is called a *G-block* if $|B| > 1$ and for each $g \in G$, $g(B) = B$ or $g(B) \bigcap B = \emptyset$.

Definition. The group G is *imprimitive* in its action on the set X if G-blocks exist.

Definition.(See [3].) Any group containing a *Burnside group* as a regular subgroup must either be doubly transitive, or imprimitive.

We introduce a powerful theorem of Burnside's:

Theorem 2.2 (Burnside; see [3].) *Every cyclic group of composite order is a Burnside group.*

Since the group P contains the regular subgroup $\mathbf{Z}_{p^n}$ which is cyclic of order p^n where $n \geq 2$, this means that P is either doubly transitive, or imprimitive. However, by the orbit-stabilizer theorem, the order of P is divisible by the lengths of the orbits of any stabilizing subgroup. If P were doubly transitive, this would mean that the orbit remaining when a single element is fixed, has length $p^n - 1$. Since $n \geq 2$, this does not divide the order of P, which is a power of p. So we may assume that P is imprimitive.

This permits us to employ induction. The inductive step is essentially to consider the action of P both within each P-block (which is a set on p^k vertices, where $k < n$), and upon the P-blocks (of which there are p^{n-k} and $n - k < n$).

The details of the proof will appear elsewhere.

3 Condition 4 is not equivalent

Having proven that any digraph which is isomorphic to Cayley digraphs on both $\mathbf{Z}_{p^n}$ and $(\mathbf{Z}_p)^n$, is isomorphic to Cayley digraphs on every other abelian group of order p^n, it seems reasonable to ask if we can find other collections from the abelian groups of order p^n for which this is true.

However, it is not hard to find a graph which is isomorphic to Cayley digraphs on every abelian group of order p^n except $\mathbf{Z}_{p^n}$. This counterexample is formed by taking p copies of the complete directed graph on p^{n-1} vertices. Since this digraph is a Cayley digraph on every abelian group G' of order p^{n-1}, we can label the vertices of each complete digraph according to one of the cosets of G' in $\mathbf{Z}_p \times G'$, so that the full digraph is a Cayley digraph on $\mathbf{Z}_p \times G'$. Now add the arcs between copies of the complete digraph, which correspond to adding the element $(1, 0, \ldots, 0)$ to the symbol set S of the Cayley digraph on $\mathbf{Z}_p \times G'$. This produces p^{n-1} disjoint directed cycles of length p among the complete subdigraphs. It is clear from the construction that the resultant digraph is a Cayley digraph on every abelian group of order p^n except perhaps $\mathbf{Z}_{p^n}$. However, since it does not satisfy the conditions in the main result of this paper, it cannot be a Cayley digraph on $\mathbf{Z}_{p^n}$.

References

[1] Anne Joseph, "The isomorphism problem for Cayley digraphs on groups of prime-squared order," *Discrete Mathematics* 141 (1995), pp. 173-183.

[2] G. Sabidussi, "On a class of fixed-point-free graphs," *Proc. Amer. Math. Soc.* 9 (1958), pp. 800-804.

[3] H. Wielandt (trans. by R. Bercov), *Finite Permutation Groups.* Academic Press, New York, 1964, pp. 64-65.