Hamilton paths in vertex-transitive graphs

Brian Alspach

School of Mathematical and Physical Sciences

The University of Newcastle

The investigation of Hamilton paths and Hamilton cycles in vertex-transitive graphs began in earnest following Lovász's presentation, at the Calgary International Combinatorics Conference in 1969, of the problem asking whether every connected vertex-transitive graph has a Hamilton path.

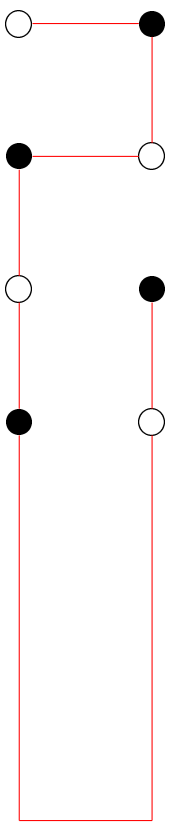
A graph is Hamilton-connected when one can find Hamilton paths with arbitrary pairs of terminal vertices. Similarly, a bipartite graph with equicardinal parts is Hamilton-laceable when one can find Hamilton paths with arbitrary pairs of terminal vertices from distinct parts.

Let $\mathbb{F}$ be a family of graphs. We say that $\mathbb{F}$ is H^*-connected if every bipartite graph in $\mathbb{F}$ is Hamilton-laceable and every non-bipartite graph in $\mathbb{F}$ is Hamilton-connected.

The result that started the thread under discussion was presented at the Australasian Combinatorics Conference in Geelong in 1979. It was published in the proceedings of that conference, thereby, delaying for some time the recognition it deserved.

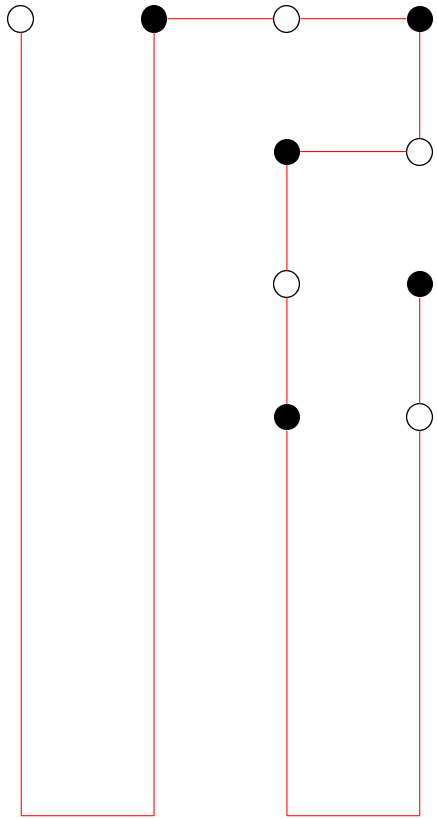
Theorem 1 (Chen-Quimpo) *The family of connected Cayley graphs of valency at least 3 on abelian groups is H^*-connected.*

Here is a sketch of their somewhat arduous proof.

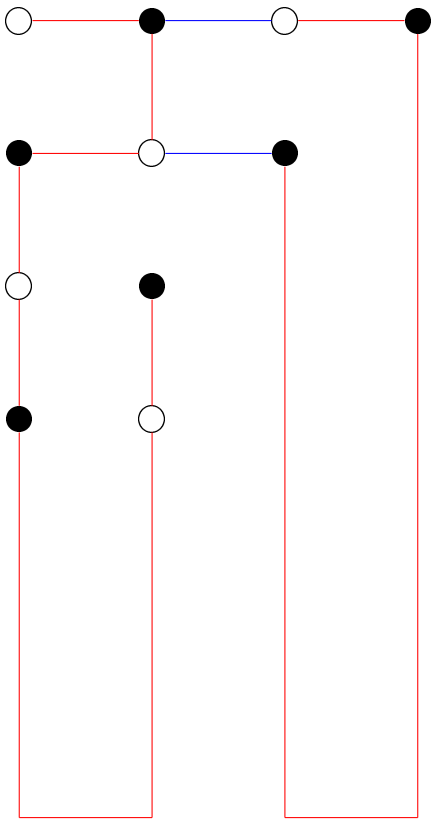


First one shows there is a Hamilton path from any corner vertex to any vertex of the other color in $P_2 \square P_k$, where P_m denotes the path of order m. Here is a typical example.

One way of juxtaposing two groups of two columns to get a Hamilton path from a corner vertex to another vertex.

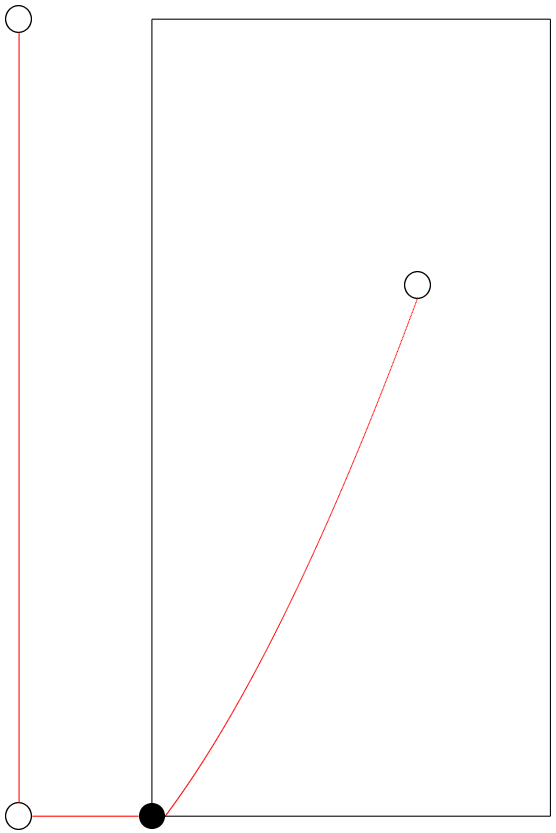


Another way of juxtaposing two groups of two columns to get a Hamilton path from a corner vertex to another vertex. Do an exchange using the two blue edges.

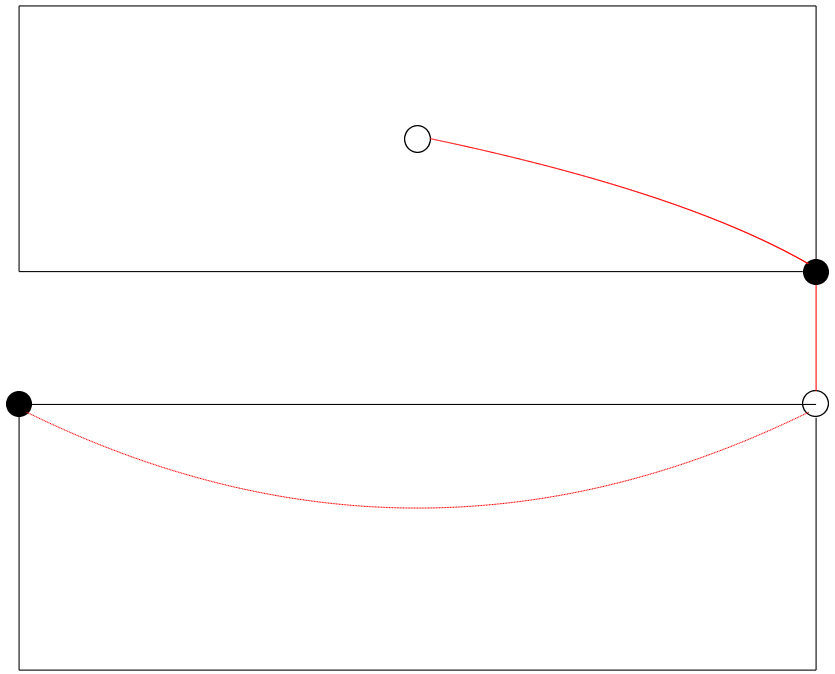


Using the ingredients shown in the preceding slides, it is not difficult to show that there is a Hamilton path from any corner vertex of $P_k \square P_\ell$ to any vertex of the other color as long as not both k and ℓ are odd.

Given $P_k \square P_\ell$, where both k and ℓ are odd, we can get a Hamilton path from any corner vertex to vertex of the same color as shown in the figure.

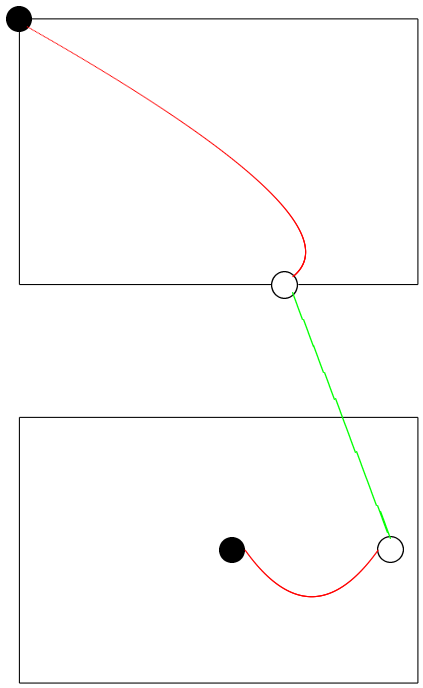


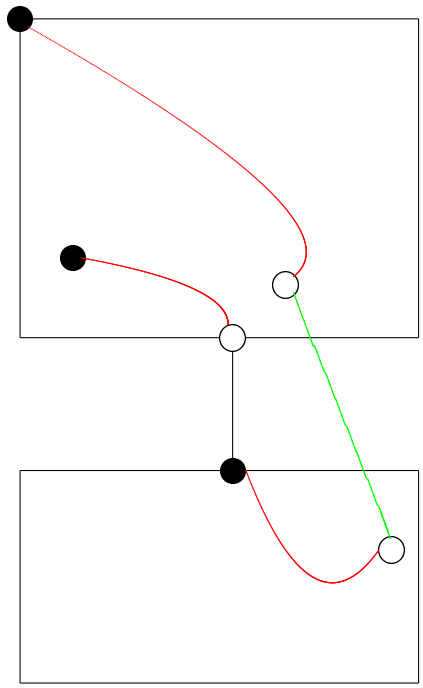
We now know there is a Hamilton path from any corner vertex of $P_k \square P_\ell$ to any vertex of opposite color when $k\ell$ is even, or to any vertex of the same color when $k\ell$ is odd. The following figure illustrates how we extend that to any vertex on the boundary.



Once we can find a Hamilton path from any vertex on the boundary to any target vertex, it is easy to split up the graph to get Hamilton paths joining any two appropriate vertices in the bipartite graph $P_k \Box P_\ell$.

Considering the case that $k\ell$ is even, the following figure indicates how one finds a Hamilton path joining two vertices of the same color when the Cayley graph is not bipartite.





They have to consider several special cases but for the most part they use spanning subgraphs of the form $P_k \Box P_\ell$ to prove their result.

Generalized Petersen graphs arise in the proof of the next result. Let me remind you how they are defined. The generalized Petersen graph GP(n, k) has vertex set $u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}$ and edge set $\{u_i v_i, u_i u_{i+1}, v_i v_{i+k} : 0 \leq i \leq n-1\}$, where the subscript arithmetic is carried out modulo n.

Note that GP$(5, 2)$ is the Petersen graph. Also note that GP$(n, 1)$ is isomorphic to the cartesian product of an n-cycle and K_2.

Even though most generalized Petersen graphs are not vertex-transitive, they are useful in the study of Hamilton paths and cycles in vertex-transitive graphs because when one has a semi-regular automorphism σ of a graph X, the subgraph induced on the vertices of two of the cycles comprising σ often contains a spanning generalized Petersen graph.

A Hamiltonian group is a non-abelian group G for which every subgroup of G is normal. It is known that G is Hamiltonian if and only if G is of the form $Q \times H_1 \times H_2$, where Q is the quaternions, H_1 is an odd order abelian group, and H_2 is an abelian group of exponent 2.

Theorem 2 (Alspach-Qin, 2001) *The family of connected Cayley graphs of valency at least 3 on Hamilton groups is H^*-connected.*

Here is a sketch of the proof of the preceding result. One building block is the fact that GP$(4m, 2m-1)$ is Hamilton-laceable for all integers $m \geq 1$. It is easy to see that GP$(4m, 2m-1)$ is bipartite so that Hamilton-laceable is the most we can hope for.

Also, GP$(4m, 2m-1)$ itself is a Cayley graph because $(2m-1)^2 \equiv$ 1(mod 4m). This implies that the mapping τ defined by

$$\begin{aligned} \tau(u_i) &= v_{i(2m-1)} \\ \tau(v_i) &= u_{i(2m-1)} \end{aligned}$$

is an automorphism of GP$(4m, 2m-1)$ and satisfies $\tau^2 = 1$. Furthermore, if we let ρ denote the automorphism

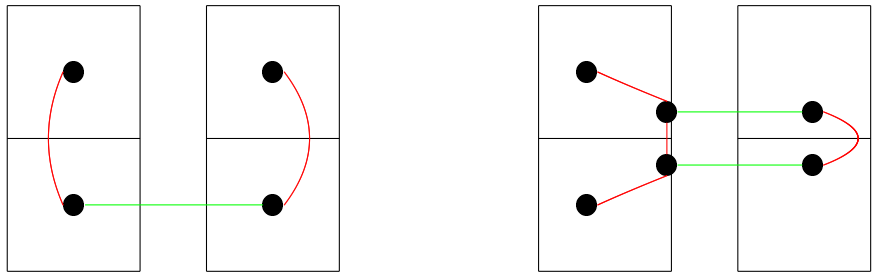
$$(u_0\ u_1\ \cdots\ u_{4m-1})(v_0\ v_1\ \cdots\ v_{4m-1}),$$

then $\tau\rho\tau = \rho^{2m-1}$. Thus, $\langle \tau, \rho \rangle$ is a regular subgroup of Aut(GP$(4m, 2m-1)$).

So to show that GP$(4m, 2m-1)$ is Hamilton-laceable, it suffices to find a Hamilton path from u_0 to u_i, i odd, and to v_j, j even. It turns out there are five cases all of which are easy to do.

Another ingredient frequently used for these kinds of results is the following lemma.

Lemma 1 *Let X be a graph formed by two vertex-disjoint subgraphs X_1 and X_2 and a perfect matching joining vertices of X_1 to vertices of X_2.*
If both X_1 and X_2 are Hamilton-connected, then X also is Hamilton-connected;
and if both X_1 and X_2 are Hamilton-laceable, then X also is Hamilton-laceable.



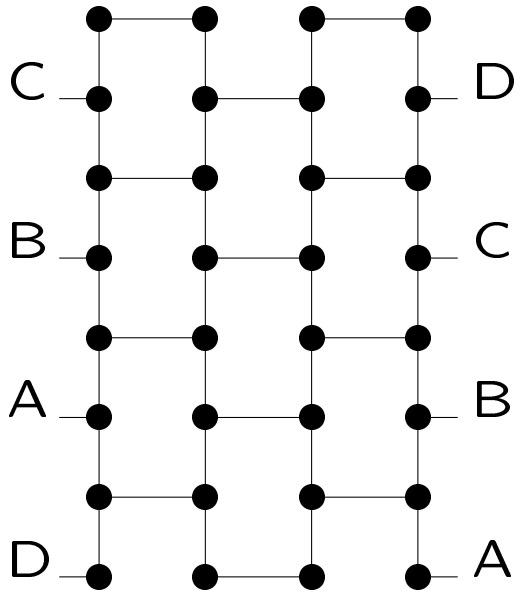
To prove the theorem about hamiltonian groups, we induct on the order of the group. The beginning of the induction consists of looking at the Cayley graphs on the quaternions.

Another result in this direction is the following theorem.

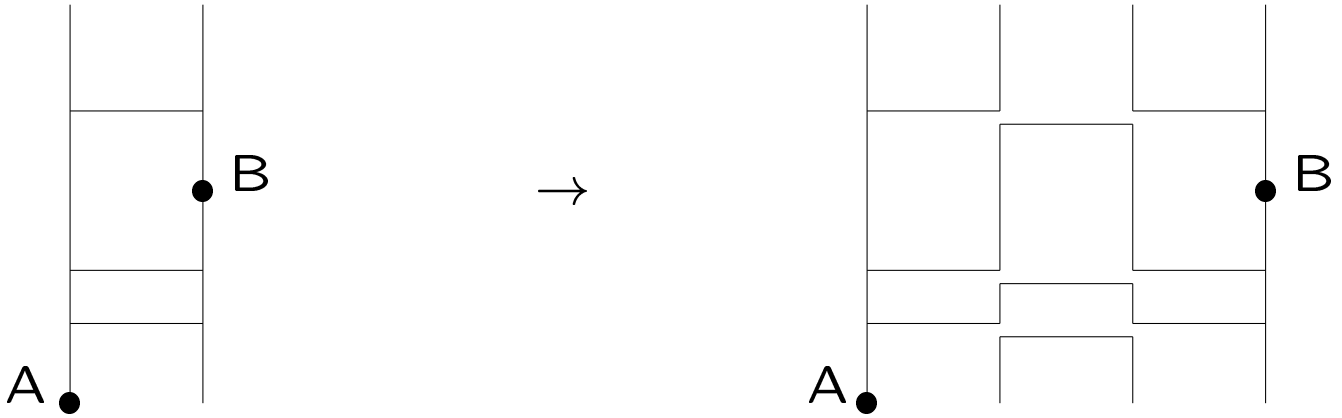
Theorem 3 (Alspach-Dean) *The family of connected Cayley graphs of valency at least 3 on generalized dihedral groups whose orders are a multiple of 4 is H^*-connected.*

As in the two previous theorems, the work to establish an induction basis is the most interesting part of the proof.

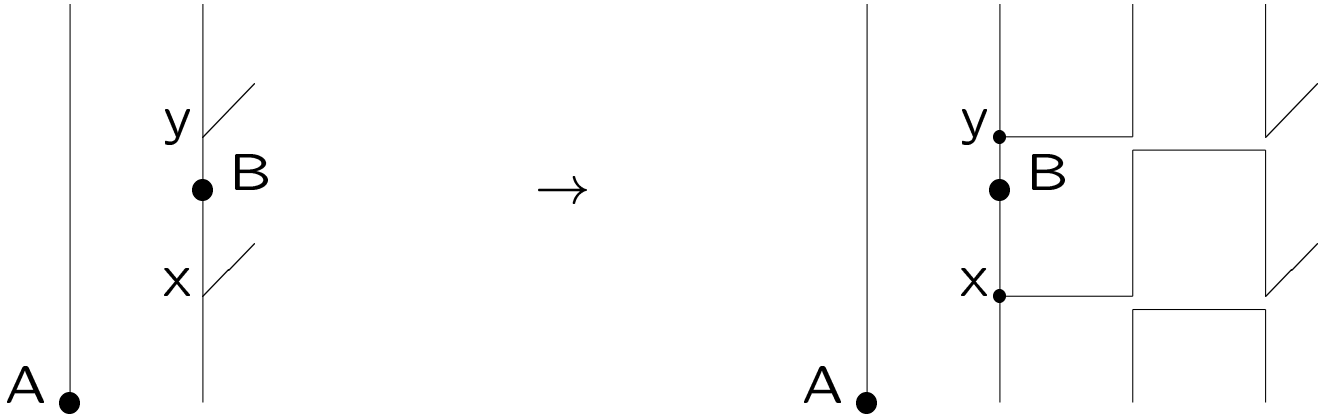
The honeycomb toroidal graph HTG(m, n, k) has vertices $u_{i,j}$, $0 \leq i \leq m-1$, $0 \leq j \leq n-1$, where $n \geq 4$ is even, and $0 \leq k < n$, and edges $u_{i,j}u_{i,j\pm1}$, $u_{i,j}u_{i+1,j}$ when $i+j$ is odd and $i \neq m-1$, and $u_{m-1,j}u_{0,j+k}$ where both $m+j$ and $m+k$ are even. Below is HTG$(4, 8, 2)$.



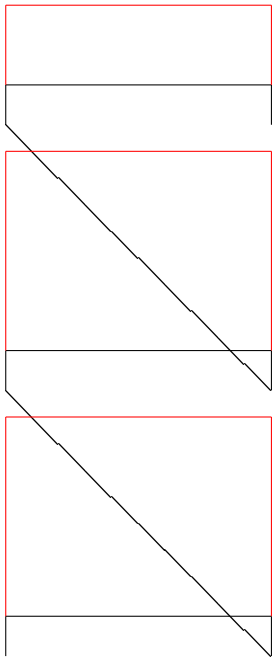
The honeycomb toroidal graphs are Hamilton-laceable when m is even. Here is a sketch of the proof. The first figure shows how to insert two new columns between the initial vertex and terminal vertex when in different columns.



The figure here shows how to insert two new columns to the right of a terminal vertex in the second column.



The preceding plus another subcase establish that it is sufficient to do the problem when $m = 2$. The next picture indicates how we solve the problem for $m = 2$.



The proof of the theorem proceeds by induction on the valency of the graph. Letting the group be $D(H)$, where H is the abelian subgroup of index 2, when the valency is 3 and the three elements of the connection set are involutions not belonging to H, the Cayley graph turns out to be a honeycomb toroidal graph with m even when the order of $D(H)$ is divisible by 4. This is the essential reason for that hypothesis. The problem for the other case is that no one has shown the honeycomb toroidal graphs are Hamilton-laceable when m is odd.

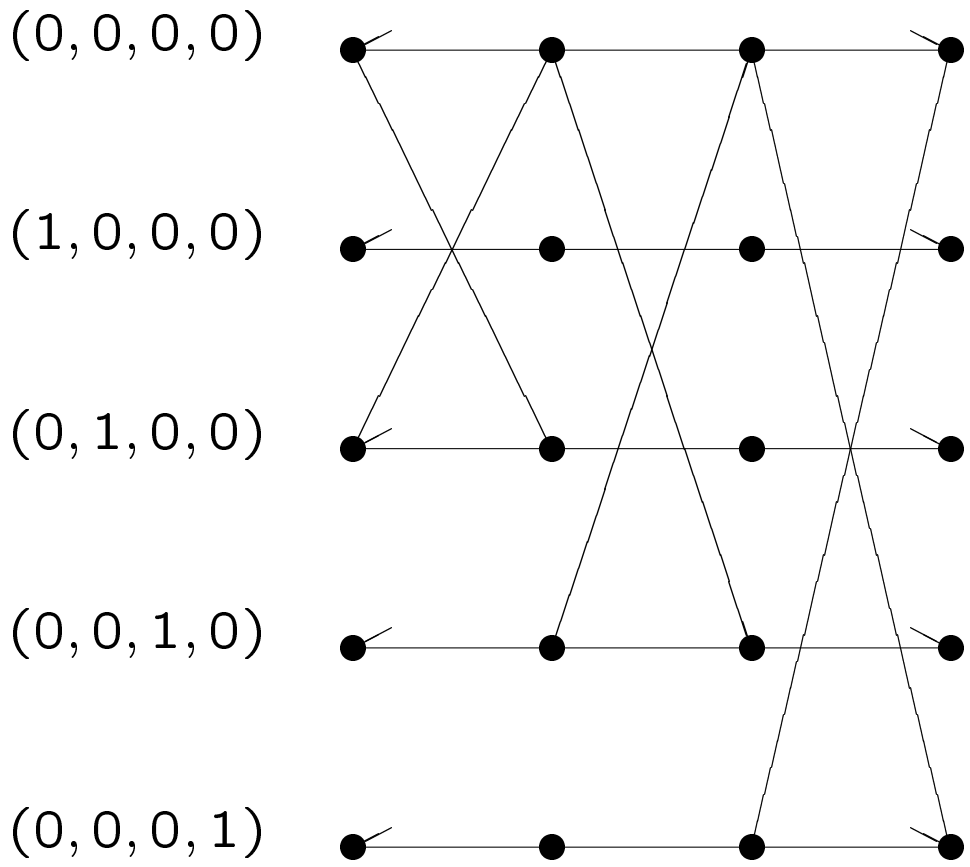
There are some results for special classes of vertex-transitive graphs. These classes have been studied mainly by computer scientists for a variety of reasons. We turn our attention to these graphs.

The groups for which we have been able to say something are either abelian or close, in some sense, to being abelian. I always believed it would be difficult to obtain results for Cayley graphs on groups that are far from being abelian.

The n-dimensional cube, denoted Q_n, actually is used in computer architecture but has some properties that are deemed problematic. In an attempt to produce graphs that might behave better for applications, there have been alternatives suggested that are based on Q_n. We look at the butterfly graphs.

The n-dimensional butterfly graph has the same vertex set as the cube-connected cycle with different adjacencies. The vertex $(a_1, a_2, \ldots, a_n; b)$ is adjacent to $(a_1, a_2, \ldots, a_n; b-1)$ and $(a_1, a_2, \ldots, a_n; b+1)$ so that we are replacing each vertex of Q_n by an n-cycle. The vertex $(a_1, a_2, \ldots, a_n; b)$ also is adjacent to the two vertices obtained by replacing b with $b-1$ and complementing the bit in the $b-1$ position, and replacing b with $b+1$ and complementing the bit in the $b+1$ position. We obtain a tetravalent graph in this case. This graph was introduced by Akers and Krishnamurthy in 1989.

Here is a picture illustrating the situation.



Some neighbors of $(0, 0, 0, 0)$-vertices in butterfly graph

Theorem 4 (Wong, 1995) *The butterfly graphs form an H^*-connected family.*

There are two families of Cayley graphs on the symmetric group with which we complete this talk. Compton and Williamson expended a lot of effort in proving that the Cayley graph on S_n with connection set consisting of an n-cycle and a particular transposition is hamiltonian. Thus, it might be surprising to some that one can say something reasonable about Hamilton-laceability for some families of Cayley graphs on S_n. Of course, Compton and Williamson were working with a trivalent graph and the graphs we are about to consider have larger valency.

Consider the symmetric group S_n. We restrict ourselves to using transpositions for the connection set. Let T be a set of transpositions from S_n. Let $A(T)$ denote an auxiliary graph with vertex set $\{1, 2, 3, \ldots, n\}$ and an edge joining i and j if and only if the transposition $(i\ j) \in T$.

The following theorem is well known.

Theorem 5 *The Cayley graph* $\mathrm{cay}(S_n; T)$*, where* T *is a set of transpositions from* S_n*, is connected if and only if the auxiliary graph* $A(T)$ *is connected.*

An n-dimensional star graph is a Cayley graph cay$(S_n; T)$, where T consists of transpositions from S_n such that the auxiliary graph $A(T)$ is a star, that is, $K_{1,n-1}$. Because all of these are isomorphic, we can talk about the n-dimensional star graph.

Hsieh, Chen, and Ho (2000) proved that the star graph is Hamilton-laceable. The following more general result was then proved later.

Theorem 6 (Araki, 2006) *A connected Cayley graph on S_n with connection set comprised of transpositions is Hamilton-laceable.*

The derangements graph is the Cayley graph on the symmetric group S_n with connection set all permutations with no fixed points. In other words, two vertices corresponding to two distinct permutations are adjacent if and only if they are derangements of each other (in range notation). This graph is Hamilton-connected because the valency is big and there is a result by Jung for regular graphs that implies the graph is Hamilton-connected.

Rasmussen and Savage refined the preceding result considerably. For k satisfying $2 \leq k \leq n$, let D_k denote all the permutations in S_n that move k successive elements $i, i+1, \ldots, i+k-1$ and fix the remaining $n-k$ elements.

Theorem 7 (Rasmussen and Savage, 1994) *The Cayley graphs on S_n with connection set D_k are Hamilton-connected for $4 \leq k \leq n$.*