Graph Coverings and Lifting Techniques

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In this talk, I shall present some methods for constructing voltage graphs by combinatorial tools and by group theoretical tools, and show a linear criteria for liftings of automorphisms when the covering transformation group is elementary abelian. As an application, I shall show a classification of 2-arc-transitive regular covers of complete graphs with the covering transformation group Z_p^3 and a classification of 2-arc-transitive Cayley graphs on dihedral groups.

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1 Voltage graphs and lifting

Graph Covering: A graph X is called a *covering* of a graph Y with the projection $p : X \to Y$ if there is a surjection $p : V(X) \to V(Y)$ such that $p|_{N(x)} : N(x) \to N(y)$ is a bijection for any $y \in V(Y)$ and $x \in p^{-1}(y)$.

X: *Covering graph*; Y: *base graph*;

Vertex fibre: $p^{-1}(v)$, $v \in V(Y)$;

Edge fibre: $p^{-1}(e)$, $e \in E(Y)$;

Fibre-preserving automorphism $\sigma \in \mathrm{Aut}\,(X)$: maps a fibre to a fibre

Covering transformation group K:

$$K = \{\sigma \in \mathrm{Aut}\,(X) \mid \sigma fix\, every\, fibre\, setwise.\}$$

Regular covering: if K acts regularly on each fibre (X is connected)

Voltage assignment f: graph Y, finite group K

a function $f : A(Y) \to K$ s. t. $f_{u,v} = {f_{v,u}}^{-1}$ for each $(u, v) \in A(Y)$.

Voltage graph $Y \times_f K$: vertex set $V(Y) \times K$, arc-set $\{((u, g), (v, f_{u,v}g)) \mid (u, v) \in A(Y),\ g \in K\}$.

Remark:

1. Voltage graph $Y \times_f K$ is a covering of Y;

2. Each connected regular covering can be reconstructed by a voltage graph.

Lifting: $\alpha \in \mathrm{Aut}\,(Y)$ *lifts* to an automorphism $\overline{a} \in \mathrm{Aut}\,(X)$ if $\alpha p = p\overline{a}$.

Question 1.1 *Given a graph Y, a group K and $H \leq$ $\mathrm{Aut}\,(Y)$, find all the connected regular coverings $Y \times_f K$ on which H lifts.*

2 Constructing voltage graphs by combinatorial methods

Lifting criterion–Topological graph theory version:

A. Malnič, Group actions, coverings and lifts of automorphisms, *Discrete Math.* **182** (1998), 203-218.

Theorem 2.1 *Let* $X = Y \times_f K$ *be a regular covering. Then* $\alpha \in \mathrm{Aut}\,(Y)$ *lifts if and only if, for each closed walk* W *in* Y*, we have* $f_{W^\alpha} = 1$ *iff* $f_W = 1$*.*

S.F. Du, J.H.Kwak and M.Y.Xu, On 2-arc-transitive covers of complete graphs with covering transformation group Z_p^3, *J. Combin. Theory, B* **93** (2005), 73–93.

Theorem 2.2 *Let* $X = Y \times_f K$ *be a connected regular cover of a graph* Y*, where* K *is abelian, If* $\alpha \in \mathrm{Aut}\,Y$ *is an automorphism one of whose liftings* $\tilde{\alpha}$ *centralizes* K, *then* $f_{W^\alpha} = f_W$ *for any closed* W *of* Y*.*

Example 2.3 *Find all the regular coverings* $X = K_5 \times_f Z_p^3$ *such that* $A_5 \leq \mathrm{Aut}\,(K_5) = S_5$ *lifts.*

Solution: Let $V(K_5) = \{0, 1, 2, 3, 4\}$ and let

$K = (V(3, p), +)$,where $p = 5$ or $p = \pm 1 (\mathrm{mod}\ 10)$. Define

$X(p) = K_5 \times_f K$ as follows:

$f_{0,j} = (0, 0, 0)$ for $1 \leq j \leq 4$,
$f_{1,2} = (1, 0, 0)$,
$f_{1,3} = (0, 1, 0)$,
$f_{2,3} = (0, 0, 1)$,
$f_{1,4} = (a, b, c)$,
$f_{2,4} = (-b, -c, a)$
$f_{3,4} = (c, -a, -b)$,
where $a = \frac{1+\sqrt{5}}{4}$, $b = \frac{1-\sqrt{5}}{4}$ and $c = \frac{\sqrt{5}}{2}$.

3 Linear criteria for liftings of automorphisms for $K = Z_p^n$

S.F. Du, J.H. Kwak and M.Y. Xu, Linear criteria for lifting of automorphisms in elementary abelian regular coverings, *Linear Alegebra and Its Applications*, 373, 101-119(2003).

Algorithm:

(1st) Choose a fixed spanning tree T in Γ and write down the arcs in $A^+(E_0)$, $A^+(E_1)$ and $A^+(E_2)$ in a certain order so that $\Phi_0 = \mathbf{0}$, $\Phi_1 = I_{n\times n}$ and $\Phi_2 = M$.

(2nd) Calculate the incidence matrix P for the fundamental cycles of Γ with respect to T.

(3rd) Assume that the voltage generating matrix $M = (a_{ij})_{m\times n}$, where the entries a_{ij} are unknowns. Let $\Delta = ((-M, I_{m\times m})P, -M, I_{m\times m})$, whose columns are indexed by the arcs in $A^+(E_0)$, $A^+(E_1), A^+(E_2)$ according to the given order. We call the matrix Δ the *discriminant matrix* for a lift of ϕ. For convenience, we write $\Delta_0 = (-M, I_{m\times m})P$, $\Delta_1 = -M$ and $\Delta_2 = I_{m\times m}$, so that $\Delta = (\Delta_0, \Delta_1, \Delta_2)$, as a block matrix.

(4th) Let $\Delta = (\cdots, \mathbf{c}_{i,j}, \cdots)$, where $c_{i,j}$ is the column indexed by $(i,j) \in A^+(\Gamma)$. For a given $\sigma \in \mathrm{Aut}\,(\Gamma)$, let $\mathbf{c}^{\sigma}_{i,j} = \mathbf{c}_{i^{\sigma^{-1}}, j^{\sigma^{-1}}}$, where we assume that $\mathbf{c}_{i,j} = -\mathbf{c}_{j,i}$

for any arc (i,j). Let $\Delta^\sigma = (\cdots, \mathbf{c}^\sigma_{i,j}, \cdots)$ for any $(i,j) \in A^+(\Gamma)$, and let $(\Delta^\sigma)_0$, $(\Delta^\sigma)_1$ and $(\Delta^\sigma)_2$ denote the first, the second and the third blocks of the matrix Δ^σ respectively, as before. Then one can say that

$$\sigma \text{ can be lifted } \iff (\Delta^\sigma)_1 + (\Delta^\sigma)_2 M = \mathbf{0}. \quad (1)$$

Problem 3.1 *Hope realize this Algorithm on the computer.*

Regular covering of the Petersen graph with covering transformation group Z_p^n

Theorem 3.2 *Let* $\widetilde{\Gamma}$ *be a connected regular covering of the Petersen graph* Γ *whose covering transformation group is isomorphic to* Z_p^n *and whose fibre-preserving automorphism subgroup* G *acts arc-transitively on* $\widetilde{\Gamma}$*. Then,* $\widetilde{\Gamma}$ *is isomorphic to one of the graphs in (i)-(iv) listed as follows. Moreover,*

(1) *for the graphs* $X(2,1)$, $X(2,2)$, $X(5,3)$ *and* $X(p,6)$, $\mathrm{Aut}\,(\Gamma) \cong S_5$ *can be lifted and so* G *acts* 3*-arc-transitively.*

(2) *For the graphs* $X'(2,1)$ *and* $X(p,3)$ *for* $p \equiv \pm 1 (\mathrm{mod}\ 10)$, *the subgroup isomorphic to* A_5 *of* $\mathrm{Aut}\,(\Gamma)$ *can be lifted but* $\mathrm{Aut}\,(\Gamma)$ *cannot, and so* G *acts* 2*-arc but not* 3*-arc-transitively.*

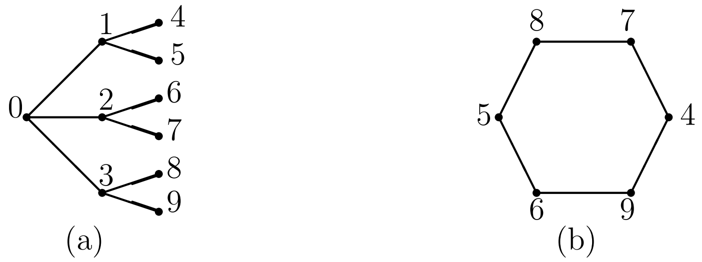

Figure 1: (a): the spanning tree T of Γ; (b): the induced subgraph $\Gamma(V_1)$.

(i) $X(2,1) := \Gamma \times_\phi Z_2$, where $\phi_{i,j} = 1$ for any cotree arc (i,j).
$X'(2,1) := \Gamma \times_\phi Z_2$, where $\phi_{5,8} = \phi_{6,9} = \phi_{4,7} = 1$ and $\phi_{5,6} = \phi_{4,9} = \phi_{7,8} = 0$.

(ii) $X(2,2) := \Gamma \times_\phi Z_2^2$, where $\phi_{5,8} = \phi_{6,9} = \phi_{4,7} = (1,0)$ and $\phi_{5,6} = \phi_{4,9} = \phi_{7,8} = (0,1)$.

(iii) $X(p,3) := \Gamma \times_\phi Z_p^3$, where $p = 5$ or $p \equiv \pm 1 (\mathrm{mod}\ 10)$, and $((\phi_{5,8})^t, (\phi_{6,9})^t, (\phi_{4,7})^t) = I_{3\times 3}$, $\phi_{5,6} = (1, -1, \frac{1+\sqrt{5}}{2})$, $\phi_{4,9} = (\frac{1+\sqrt{5}}{2}, 1, 1)$ and $\phi_{7,8} = (1, \frac{1+\sqrt{5}}{2}, -1)$.

(iv) $X(p,6) := \Gamma \times_\phi Z_p^6$, where
$((\phi_{5,8})^t, (\phi_{5,6})^t, (\phi_{6,9})^t, (\phi_{4,9})^t, (\phi_{4,7})^t, (\phi_{7,8})^t) = I_{6\times 6}$.

Malnic, Aleksander; Potocnik, Primoz. Invariant subspaces, duality, and covers of the Petersen graph, *European J. Combin.* 27 (2006), no. 6, 971–989.

4 Constructing voltage graphs by group theoretical methods

Let $X = Y \times_f K$ and let $G \leq \mathrm{Aut}\,(Y)$. Suppose that G lifts. Then $A/K \cong G$.

General Idea:

Step 1: Determine the group A by the group theoretical tools;

Step 2: Determine the permutation representations of A relative to all the possible subgroups H (point stabilizer);

Step 3: Determine the corresponding suborbit structure of the above representations so that obtain the coset graphs;

Step 4: Find the voltage assignment from these coset graphs.

Example 4.1 *Let* $Y = K_{1+p^k}$ *where* $V(Y) = PG(1, p^k) = GF(p^k) \cup \{\infty\}$ *and let* $K = (V(3, p), +)$. *Find all the regular coverings* $X = Y \times_f K$ *such that* $\mathrm{PGL}(2, p^k) \leq \mathrm{Aut}\,(Y)$ *lifts.*

Solution: Define $X(p^k) =: K_{1+p^k} \times_f Z_p^3$ as follows:

$f_{\infty,j} = (0, 1, 2j)$,

$f_{i,j} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ for all $i \neq j$ in $GF(p^k)$.

Example 4.2 *Find all the regular coverings* $X = K_8 \times_f Z_p^3$ *such that* $\mathrm{AGL}(3,2) \cong Z_2^3 \rtimes \mathrm{GL}(3,2) \leq \mathrm{Aut}\,(K_8) = S_8$ *lifts.*

Let

$V(K_{2^m}) = V(m, 2)$,

$\Omega = PG(m-1, 2)$, (identify with $V(m,2) \setminus \{0\}$.)

$V = V(\Omega)$: the linear space of all the characteristic functions of Ω, $(2^m - 1)$-dimensional.

V_1 and V_2: the subspaces of $V(\Omega)$ generated by the characteristic functions of all 1-dimensional subspaces and of all 2-dimensional subspaces of $PG(m-1, 2)$, respectively.

$K = (V_1/V_2, +)$: the additive group of quotient space V_1/V_2 which is $\frac{m(m-1)}{2}$-dimensional.

Define $X(m) = K_{2^m} \times_f K$ $(m \geq 3)$ as follows:

$f_{0,j} = \overline{0} := V_2$,

$f_{i,j} = \overline{\chi}_{\{i,j,i+j\}} := \chi_{\{i,j,i+j\}} + V_2$ for all $i \neq j$ in Ω.

Solution: p must be 2 and the covering graph is exactly $X(3)$.

5 Classifying 2-arc-transitive graphs

Let X be a simple graph with vertex-set $V(X)$, edge-set $E(X)$, arc-set $A(X)$, automorphism group $\mathrm{Aut}\,(X)$. For $s \geq 1$, an s-arc is an sequence $(v_0, v_1 \cdots, v_s\}$ for $(v_{i-1}, v_i) \in A(X)$ and $v_{i-1} \neq v_{i+1}$. Then X is said to be *vertex-, edge-, s-arc- transitive,* respectively, if $\mathrm{Aut}\,(X)$ acts transitively on $V(X)$, $E(X)$ and s-arcs of X.

Theorem 5.1 *(Tutte 1947, Wong, 1967) There exist no finite s-arc-transitive cubic graphs for* $s \geq 6$. *The point-stabilizers are respectively* Z_3, S_3, D_{12}, S_4 *and* $S_4 \times Z_2$, *corresponding to s=1, 2, 3, 4, and 5.*

Theorem 5.2 *(Weiss, 1981) There exist no finite s-arc-transitive graphs for* $s = 6$ *and* $s \geq 8$.

Given any finite permutation group with enough large order, one can construct some finite 1-arc-transitive graphs (symmetric graph). Therefore, we may pay attention to the classification of finite 2-arc-transitive graphs.

Praeger's Reduction Theorem

C.E. Praeger, An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, *J. London Math. Soc.* (2)**47**(1993), 227-239.

Every finite connected 2-arc-transitive graphs is:

(1) *Quasiprimitive Type:* every non-trivial normal subgroup of $\mathrm{Aut}\,X$ acts transitively on $V(X)$,

Bipartite Type: every non-trivial normal subgroup of $\mathrm{Aut}\,X$ has at most two orbits on $V(X)$ and at least one of normal subgroups of $\mathrm{Aut}\,X$ has exactly two orbits on $V(X)$.

(2) *Covering Type:* regular covers of graphs in (1).

A permutation group is said to be *quasiprimitive* if every nontrivial normal subgroup of G is transitive.

In first case of (1), Aut X acts quasiprimitively on $V(X)$.

R.W.Baddeley, Two-arc transitive graphs and twisted wreath products, *J. Alg. Combin.* **2** (1993), 215–237.

A.A. Ivanov and C.E. Praeger, On finite affine 2-arc-transitive graphs, *Europ. J. Combin.* **14** (1993), 421–444.

X.G. Fang and C.E. Praeger, On graphs admitting arc-transitive actions of almost simple groups, *J. Algebra* **205** (1998), 37-52.

X.G. Fang and C.E. Praeger, Finite two-arc-transitive graphs admitting a Suzuki simple group, *Comm. Algebra* **27**(1999), 3727-3754.

X.G. Fang and C.E. Praeger, Finite two-arc-transitive graphs admitting a Ree simple group, *Comm. Algebra* **27**(1999), 3755-3769.

C. H. Li, The finite vertex-primitive and vertex-biprimitive s-transitive graphs for $s \geq 4$, *Trans. Amer. Math. Soc.* **353** (2001), 3511–3529.

In the second case of (1), X must be a bipartite graph and a reduction theorem for this case was given by Praeger (1993).

C. E. Praeger, On a reduction theorem for finite, bipartite, 2-arc-transitive graphs, *Australas J. Combin.* **7** (1993), 21-36.

There are few known results for *covering type.*

First, study minimal regular coverings, that is, K is a characteristically simple group. Then

$K \cong Z_p^n$; or
T^n for a nonabelian finite simple group T.

6 Application: Classify 2-arc-transitive regular covers of complete graphs

Problem 6.1 *Classify regular covers of complete graphs having the covering transformation group* $\mathbb{Z}_p^k$ *and whose fibre-preserving group acts* 2*-arc -transitively.*

For $k = 1, 2$:
S.F.Du, D.Marušič and A.O.Waller, On 2-arc-transitive covers of complete graphs, *J. Combin. Theory, B* **74** (1998), 276–290.

For $k = 3$:
S.F.Du, J.H.Kwak and M.Y. Xu, On 2-arc-transitive covers of complete graphs with covering transformation group Z_p^3, *J. Combin. Theory, B* **93** (2005), 73–93.

For $k \geq 4$: open

Theorem 6.2 *Let X be a connected regular cover of a complete graph K_n $(n \geq 4)$ satisfying the following two properties:* (1) *the covering transformation group is isomorphic to the elementary abelian p-group $\mathbb{Z}_p^3$, and* (2) *the group of fibre-preserving automorphisms acts* 2*-arc-transitively. Then, X is isomorphic to one of the graphs belonging to the four families listed as follows: $X_1(p) = K_4 \times_f \mathbb{Z}_p^3$, $X_2(p) = K_5 \times_f \mathbb{Z}_p^3$ for $p = 5$ or $p \equiv \pm 1 (\mathrm{mod}\ 10)$, $X_3(p) = K_{1+p} \times_f \mathbb{Z}_p^3$ for $p \geq 5$, and $X_4(3) = K_8 \times_f \mathbb{Z}_2^3$. Conversely, for each cover contained in the four families as below, the group of fibre-preserving automorphisms acts* 2*-arc-transitively.*

(1) Let Y be any connected graph with the betti number $\beta = \beta(Y) = |E(Y)| - |V(Y)| + 1$. With a fixed spanning tree Y_1 of Y, we define the cover $Y \times_f \mathbb{Z}_p^\beta$ as follows: $f_{i,j} = 0$ for $(i, j) \in A(Y_1)$ and $\{f_{i,j} \mid (i, j) \in A(Y) \backslash A(Y_1)\}$ generates $\mathbb{Z}_p^\beta$, *i.e.*, $\langle f_{i,j} \mid (i, j) \in A(Y) \backslash A(Y_1)\rangle = \mathbb{Z}_p^\beta$.

    Clearly, each cover in this family is connected and it is *homological*. In particular, by $X_1(p)$ we denote such cover $K_4 \times_f \mathbb{Z}_p^3$ for $Y = K_4$ and $\beta(K_4) = 3$.

(2) Let $V(K_5) = \{0, 1, 2, 3, 4\}$ and let K be the additive group of the 3-dimensional vector space $V(3, q)$ over the finite field $GF(q)$, where $q = p^k$ for a prime p and either $q = 5$ or $q = \pm 1 (\mathrm{mod}\ 10)$. By $X_2(q)$, we

denote the cover $K_5 \times_f K$ defined as follows: $f_{0,j} = (0,0,0)$ for $1 \le j \le 4$, $f_{1,2} = (1,0,0)$, $f_{1,3} = (0,1,0)$, $f_{2,3} = (0,0,1)$, $f_{1,4} = (a,b,c)$, $f_{2,4} = (-b,-c,a)$ and $f_{3,4} = (c,-a,-b)$, where $a = \frac{1+\sqrt{5}}{4}$, $b = \frac{1-\sqrt{5}}{4}$ and $c = \frac{\sqrt{5}}{2}$.

Let H be the subgroup of K generated by all the voltages on cotree-arcs. Then, it is easy to see that $|H| = p^3$ (resp. p^6) if $\sqrt{5} \in GF(p)$ (resp. $\sqrt{5} \in GF(q) \setminus GF(p)$). Therefore, in the case of $q = p$, the cover $X_2(p)$ is connected; and in the case of $q = p^2$ and $\sqrt{5} \in GF(q) \setminus GF(p)$, the cover $X_2(p^2)$ is isomorphic to the cover $K_5 \times_f \mathbb{Z}_p^6$ defined in (1). Moreover, the cover $X_2(q)$ is a union of connected covers isomorphic to either $X_2(p)$ or $K_5 \times_f \mathbb{Z}_p^6$ defined in (1).

(3) Let $q = p^k$, for any prime p and any natural number k, and we identify $V(K_{1+q})$ with the projective line $PG(1,q) = GF(q) \cup \{\infty\}$. Let K be the additive group of the vector space $V(3,q)$. Denote by $X_3(q)$ the cover $K_{1+q} \times_f K$ defined as follows: $f_{\infty,j} = (0,1,2j)$ and $f_{i,j} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ for all $i \ne j$ in $GF(q)$.

Let H be the subgroup of K generated by the voltages on closed walks of K_{1+q}. Then, the cover $X_3(q)$ is a union of $|K : H|$ isomorphic connected covers. In particular, if $q = p \ge 5$, then $H = K \cong \mathbb{Z}_p^3$, and in this case the cover $X_3(p)$ is connected.

(4) Let Ω be a finite set. Given a $\Delta \in P(\Omega)$, the power set of Ω, let χ_Δ denote the characteristic function of Δ, that is, if $\chi_\Delta(i) = 1$ for $i \in \Delta$ and $\chi_\Delta(i) = 0$ for $i \notin \Delta$. Then, the set $V(\Omega)$ of all characteristic functions χ_Δ, where $\Delta \in P(\Omega)$, forms a vector space over $GF(2)$ with the rule: $(a\chi_\Delta + b\chi_\Gamma)(i) = a\chi_\Delta(i) + b\chi_\Gamma(i)$ for any $a, b \in GF(2)$ and $\chi_\Delta, \chi_\Gamma \in V(\Omega)$. Clearly, a natural basis for $V(\Omega)$ is the set of characteristic functions $\chi_{\{i\}}$ for all $i \in \Omega$.

Now, for $m \geq 3$, we identify $V(K_{2^m})$ with $V(m, 2)$. Let $\Omega = PG(m-1, 2)$, the $(m-1)$-dimensional projective space over the field $GF(2)$, while we identify Ω with $V(m, 2) \setminus \{0\}$. Then, $V = V(\Omega)$ is a $(2^m - 1)$-dimensional vector space. Let V_1 and V_2 be the subspaces of $V(\Omega)$ generated by the characteristic functions of all 1-dimensional subspaces and of all 2-dimensional subspaces of $PG(m-1, 2)$, respectively. Then, the dimension of the quotient space V_1/V_2 is $\frac{m(m-1)}{2}$. Let K be the corresponding additive group of V_1/V_2. Now, for $m \geq 3$, we define the cover $X_4(m) = K_{2^m} \times_f K$ as follows: $f_{0,j} = \overline{0} := V_2$ and $f_{i,j} = \overline{\chi}_{\{i,j,i+j\}} := \chi_{\{i,j,i+j\}} + V_2$ for all $i \neq j$ in Ω.

Clearly, the cover $X_4(m)$ is connected. Let $\mathbf{1}$ be the constant characteristic function on Ω. If $m = 3$, then $V_2 = \langle \mathbf{1} \rangle$, and $V_1 = V_2 \oplus W$, where W is the unique 3-dimensional subspace of V generated by all the functions of the complements of lines in $PG(m-1,2)$.

Key point of Proof:

Let $X = K_n \times_f K$ be a connected regular cover of a complete graph K_n ($n \geq 4$) whose covering transformation group $K = Z_p^k$ and whose fibre-preserving automorphism subgroup A acts 2-arc-transitively on X.

Since A acts 2-arc-transitively on X, A/K acts 2-arc-transitively on the base graph $Y = K_n$. This forces A/K to be a 3-transitive group on $V(K_n)$.

Step 1: Determine the structure of A such that $A/K =$ 3-transitive group, we also need the structures of $\mathrm{GL}(k,p)$ (for small value k)

Step 2: For each case, determine the corresponding coset graphs as well as voltage graphs.

Problem 6.3 *Study the irreducible modular representation of 2-transitive groups.*

B. Mortimer, The modular permutation representations of the known doubly transitive groups, *Proc. London Math. Soc.* **41**(3)(1980), 1–20.

Problem 6.4 *Investigate the connections in methods between topological graph theory, group extension and group representations, and find some more efficient tools for constructing voltage graphs.*

Problem 6.5 *Classify 2-arc-transitive coverings of complete graphs with nonabelian covering transformation groups.*

Problem 6.6 *Classify 2-arc-transitive coverings of other 2-arc-transitive quasiprimitive graphs or bipartite graphs*

7 Application: Classify 2-arc-transitive dihedrant

For a Cayley graph, its automorphism group contains a vertex-regular subgroup.

Cayley graphs of cyclic and dihedral groups are called *Circulant* and *Dihedrants*, respectively.

B.Alspach, M.D.E.Conder, D.Marušič and M.Y.Xu, A classification of 2-arc-transitive circulants, *J. Alg. Combin.*, **5** (1996), 83–86.

Theorem 7.1 *2-arc-transitive circulants of valency greater than 2 are: complete graphs K_m, complete bipartite graphs $K_{m,m}$, or $K_{m,m} - mK_2$.*

The proof is combinatorial and is independent on CFSG.

D. Marušič, On 2-arc-transitivity of Cayley graphs, *J. Combin. Theory, B* **87** (2003), 162–196.

D. Marušič, Corrigendum to "On 2-arc-transitivity of Cayley graphs, J. Combin. Theory Ser. B 87 (2003) 162196," *J. Combin. Theory, B* **96** (2006), 761–764.

Theorem 7.2 *Let $n \geq 3$, and let X be a connected, 2-arc-transitive Cayley graph of a dihedral group $D = D_{2n} = \langle \rho, \tau | \rho^n = \tau^2 = (\rho\tau)^2 = 1 \rangle$ of order $2n$. Then one of the following occurs:*

(i) either X is isomorphic to $K_{n,n}$, $K_{n,n} - nK_2$ $B(H_{11})$ and $B'(H_{11})$, $B(PG(d,q))$ and $B'(PG(d,q))$,

(ii) cycles C_{2n}, complete graphs K_{2n}, and graphs K^4_{q+1}; or

(iii) X is a regular cyclic cover of a graph in (i).

S.F. Du, A. Malnič and D. Marušič, Classification of 2-arc-transitive dihedrants, *J. Combin. Theory, B*, 98(6), (2008), 1349-1372

Theorem 7.3 *Let X be a connected 2-arc-transitive Cayley graph of a dihedral group of order $2n$, where $n \geq 3$. Then*

(i) X is a basic graph and is isomorphic to one of the following graphs: C_{2n}, n a prime; K_{2n}; $K_{n,n}$; $B(H_{11})$ or $B'(H_{11})$; $B(PG(d,q))$ or $B'(PG(d,q))$,

where $n = (q^d - 1)/(q-1)$, $d \geq 2$, *and* q *is a prime power; or*

(ii) X *is isomorphic to* $K_{n,n} - nK_2$ *(*Z_2*-cover of* K_n*) or* $K_{q+1}^{2d} = Y \times_f K$*, where* $Y = K_{n,n} - nK_2$*, for* $V(Y) = U \cup W$*,*

$$\begin{array}{rcl} U & = & \mathrm{PG}(1,q) = \{\infty, 0, 1, \cdots, q-1\} \\ W & = & \{\infty', 0', 1', \cdots, (q-1)'\}. \end{array}$$

and where $ij' \in E(Y)$ *for* $i, j \in U$ *and* $i \neq j$*;* $K = \langle r \rangle \cong Z_d$*, where* d *is a divisor of* $\frac{q-1}{2}$ *when* $q \equiv 1 (\mathrm{mod}\ 4)$ *and a divisor of* $q-1$ *when* $q \equiv 3 (\mathrm{mod}\ 4)$*, respectively.*

Set

$$\begin{array}{lll} f(\infty, i') & = & f(i, \infty') = 1, \quad \text{for} \quad i \neq \infty \\ f(i, j') & = r^h & \quad \text{for} \quad i, j \neq \infty,\ i - j = \theta^h. \end{array}$$

Recall that a group H is called *Burnside group*, if any permutation group containing a regular subgroup isomorphic to H is either imprimitive or 2-transitive.

Our classification depends on:

the fact that dihedral group is a Burnside group;

classification of 2-transitive groups which depends on CFSG;

maximal subgroup structure of some simple groups;

some results and methods on Coset graphs and bi-coset graphs;

voltage graphs and lifting techniques in group theory as well as topological graphs theory.