Solutions to Homework 3 - Math 3410

1. (Page 163: # 4.77) Determine whether or not W is a subspace of $\mathbb{R}^3$ such that: (a) $a = 3b$, (b) $a \le b \le c$, (c) $ab = 0$, (d) $a+b+c = 0$, (e) $b = a^2$, (f) $a = 2b = 3c$.

   Solution (a) Since $0 = 3(0)$ we have $(0, 0, 0) \in W$.

   If $(a_1, b_1, c_1) \in W$ and $(a_2, b_2, c_2) \in W$ we have $a_1 = 3b_1$ and $a_2 = 3b_2$, so $a_1 + a_2 = 3(b_1 + b_2)$, thus $(a_1 + a_2, b_1 + b_2, c_1 + c_2) \in W$.

   If $(a, b, c) \in W$ and $k \in \mathbb{R}$, we have $a = 3b$ and so $ka = 3(kb)$. Thus $k(a, b, c) \in W$.

   Therefore by Theorem 4.2 W is a subspace of $\mathbb{R}^3$.

   (b) $(1, 2, 3) \in W$ however $-1(1, 2, 3) = (-1, -2, -3) \notin W$, so W is not closed under scalar multiplication and so it is not a subspace of $\mathbb{R}^3$.

   (c) $(0, 1, 1) \in W$ and $(1, 0, 0) \in W$, however $(0, 1, 1) + (1, 0, 0) = (1, 1, 1) \notin W$, so W is not closed under addition and thus it is not a subspace of $\mathbb{R}^3$.

   (d) Since $0 + 0 + 0 = 0$ we have $(0, 0, 0) \in W$.

   If $(a_1, b_1, c_1) \in W$ and $(a_2, b_2, c_2) \in W$ we have $a_1+b_1+c_1 = 0$ and $a_2+b_2+c_2 = 0$, so $(a_1+a_2)+(b_1+b_2)+(c_1+c_2) = 0$. This shows that $(a_1+a_2, b_1+b_2, c_1+c_2) \in W$.

   If $(a, b, c) \in W$ and $k \in \mathbb{R}$, we have $a + b + c = 0$ and so $ka + kb + kc = 0$. Thus $k(a, b, c) \in W$.

   Therefore by Theorem 4.2 W is a subspace of $\mathbb{R}^3$.

   (e) $(1, 1, 0) \in W$, however $2(1, 1, 0) = (2, 2, 0) \notin W$, so W is not closed under scalar multiplication and so it is not a subspace of $\mathbb{R}^3$.

   (f) Since $0 = 2(0) = 3(0)$ we have $(0, 0, 0) \in W$.

   If $(a_1, b_1, c_1) \in W$ and $(a_2, b_2, c_2) \in W$ we have $a_1 = 2(b_1) = 3(c_1)$ and $a_2 = 2(b_2) = 3(c_2)$, so $(a_1+a_2) = 2(b_1+b_2) = 3(c_1+c_2)$. This shows that $(a_1+a_2, b_1+b_2, c_1+c_2) \in W$.

   If $(a, b, c) \in W$ and $k \in \mathbb{R}$, we have $a = 2b = 3c$ and so $ka = 2kb = 3kc$. Thus $k(a, b, c) \in W$.

   Therefore by Theorem 4.2 W is a subspace of $\mathbb{R}^3$.

2. (Page 163: # 4.78) Let V be the vector space of n-square matrices over a field K. Show that W is a subspace of V if W consists of all matrices $A = [a_{ij}]$ that are

   (a) symmetric ($A^T = A$ or $a_{ij} = a_{ji}$), (b) (upper) triangular, (c) diagonal, (d) scalar.

   Solution (a) Since $0^T = 0$ we have $0 \in W$.

   If $A \in W$ and $B \in W$ we have $A^T = A$ and $B^T = B$, so $(A + B)^T = A^T + B^T = A + B$. This shows that $A + B \in W$.

   If $A \in W$ and $k \in K$, we have $A^T = A$ and so $(kA)^T = kA^T = kA$. Thus $kA \in W$.

   Therefore by Theorem 4.2 W is a subspace of V.

   (b) Recall that a square matrix $A = [a_{ij}]$ is upper triangular if $a_{ij} = 0$ for $i > j$.

   Since 0 is upper triangular we have $0 \in W$.

   If $A = [a_{ij}] \in W$ and $B = [b_{ij}] \in W$ we have $a_{ij} = b_{ij} = 0$ for $i > j$, so $a_{ij}+b_{ij} = 0$ for $i > j$. This shows that $A + B$ is upper triangular and so $A + B \in W$.

   If $A = [a_{ij}] \in W$ and $k \in K$, we have $a_{ij} = 0$ for $i > j$ and so $ka_{ij} = 0$ for $i > j$. Thus kA is upper triangular and so $kA \in W$.

Therefore by Theorem 4.2 W is a subspace of V.

(c) Recall that a square matrix $A = [a_{ij}]$ is diagonal if $a_{ij} = 0$ for $i \neq j$.

Since 0 is diagonal we have $0 \in W$.

If $A = [a_{ij}] \in W$ and $B = [b_{ij}] \in W$ we have $a_{ij} = b_{ij} = 0$ for $i \neq j$, so $a_{ij}+b_{ij} = 0$ for $i \neq j$. This shows that $A + B$ is diagonal and so $A + B \in W$.

If $A = [a_{ij}] \in W$ and $k \in K$, we have $a_{ij} = 0$ for $i \neq j$ and so $ka_{ij} = 0$ for $i \neq j$. Thus kA is diagonal and so $kA \in W$.

Therefore by Theorem 4.2 W is a subspace of V.

(d) Recall that a diagonal matrix $A = [a_{ij}]$ is called scalar if $a_{ii} = a$ for fixed $a \in K$.

Since 0 is an scalar matrix we have $0 \in W$.

If $A = [a_{ij}] \in W$ and $B = [b_{ij}] \in W$ we have $a_{ij} = b_{ij} = 0$ for $i \neq j$, $a_{ii} = a$, and $b_{ii} = b$, so $a_{ij} + b_{ij} = 0$ for $i \neq j$ and $a_{ii} + b_{ii} = a + b$. This shows that $A + B$ is an scalar matrix and so $A + B \in W$.

If $A = [a_{ij}] \in W$ and $k \in K$, we have $a_{ij} = 0$ for $i \neq j$ and $a_{ii} = a$, so $ka_{ij} = 0$ for $i \neq j$, and $ka_{ii} = ka$. Thus kA is an scalar matrix and so $kA \in W$.

Therefore by Theorem 4.2 W is a subspace of V.

3. (Page 163: # 4.79) Let $AX = B$ be a nonhomogeneous system of linear equations in n unknowns, that is, $B \neq 0$. Show that the solution set is not a subspace of K^n.

   Solution Let X_1 and X_2 be two solutions of $AX = B$. We have $AX_1 = B$ and $AX_2 = B$ and so $A(X_1 + X_2) = AX_1 + AX_2 = B + B = 2B \neq B$. (Since $B \neq 0$ we have $B \neq 2B$.) So $X_1 + X_2$ is not a solution of $AX = B$, thus the solution set of $AX = B$ is not closed under addition and so it is not a subspace of K^n.

4. (Page 163: # 4.80) Suppose U and W are subspaces of V for which $U \cup W$ is a subspace. Show that $U \subseteq W$ or $W \subseteq U$.

   Solution Suppose that $U \cup W$ is a subspace of V but $U \nsubseteq W$ and $W \nsubseteq U$. Since $U \nsubseteq W$ then there is $x \in U$ such that $x \notin W$. Similarly since $W \nsubseteq U$ there is $y \in W$ such that $y \notin U$.

   We now consider $x + y$. Since $U \cup W$ is a subspace we have $x + y \in U \cup W$. Now $x + y \in U$ or $x + y \in W$. We show that both cases result in contradictions. If $x + y \in W$, then $x = (x + y) - y \in W$ which is a contradiction since $x \notin W$. Similarly if $x + y \in U$, then $y = (x + y) - x \in U$ which is again a contradiction since $y \notin U$.

   The contradiction shows that either $U \subseteq W$ or $W \subseteq U$.