Solutions to Homework 4 - Math 3410

1. (Page 157: # 4.85) Find one vector in $\mathbb{R}^3$ that spans the intersection of U and W where U is the $x-y$ plane - that is, $U = \{(a,b,0)^T \mid a,b \in \mathbb{R}\}$ - and W is spanned by the vectors $(1,1,1)^T$ and $(1,2,3)^T$.

   Solution Note that

$$W = \{\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R}\}$$

   Now note that an element of U has its third coordinate equal to 0. Thus an element of $U \cap W$ has the form

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \alpha + \beta \\ \alpha + 2\beta \\ \alpha + 3\beta \end{pmatrix}$$

   with the additional constraint

$$\alpha + 3\beta = 0 \text{ or } \alpha = -3\beta.$$

   Therefore such a vector would be of the form

$$(-3\beta) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \beta \begin{pmatrix} -3+1 \\ -3+2 \\ -3+3 \end{pmatrix} = \beta \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

   with $\beta \in \mathbb{R}$. Thus $U \cap W$ is spanned by $\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$ and

$$U \cap W = \text{span}\{\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}\}.$$

2. (Page 157: # 4.86) Prove that span(S) is the intersection of all subspaces of V containing S.

   Solution By Theorem 4.5(ii) we know that if W is a subspace of W and $S \subseteq W$ then span$(S) \subseteq W$. It follows that S is contained in the intersection of all vector spaces containing S. Or in symbols

$$\text{span}(S) \subseteq \bigcap_{\substack{W \text{ a subspace of V} \\ S \subseteq W}} W.$$

   On the other hand, let

$$\vec{x} \in \bigcap_{\substack{W \text{ a subspace of V} \\ S \subseteq W}} W.$$

   This says that $\vec{x}$ is contained in each subspace of V containing S. But span(S) is one of these subspaces. Therefore

$$\vec{x} \in \text{span}(S).$$

It follows that

$$\bigcap_{\substack{W \text{ a subspace of V} \\ S \subseteq W}} W \subseteq \operatorname{span}(S)$$

and hence

$$\operatorname{span}(S) = \bigcap_{\substack{W \text{ a subspace of V} \\ S \subseteq W}} W.$$

3. (Page 157: # 4.87) Show that $\operatorname{span}(S) = \operatorname{span}(S \cup \{0\})$. That is by joining or deleting the zero vector from a set, we do not change the space spanned by the set.

   Solution Let $S = \{\vec{v}_1, \cdots, \vec{v}_n\}$ and $\vec{v} \in \operatorname{span}(S)$. So there are scalars $a_1, \cdots, a_n$ such that

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n.$$

   It is clear that

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n + b\vec{0}.$$

   This shows that $\vec{v} \in \operatorname{span}(S \cup \{0\})$. In other words $\operatorname{span}(S) \subseteq \operatorname{span}(S \cup \{\vec{0}\})$.

   Now suppose that $\vec{v} \in \operatorname{span}(S \cup \{\vec{0}\})$, so there are scalars $a_1, \cdots, a_n$, and b such that

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n + b\vec{0}.$$

   So

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n.$$

   This shows that $\vec{v} \in \operatorname{span}(S)$. In other words $\operatorname{span}(S \cup \{\vec{0}\}) \subseteq \operatorname{span}(S)$.

   From the previous two paragraphs we have $\operatorname{span}(S) = \operatorname{span}(S \cup \{0\})$.

4. (Page 157: # 4.88(a)) Show that if $S \subseteq T$, then $\operatorname{span}(S) \subseteq \operatorname{span}(T)$.

   Solution Let $|S| = n$ and $|T| = n + j$ where $n \in \mathbb{N}$ and $j \geq 0$. We have

$$S = \{\vec{v}_1, \ldots, \vec{v}_n\}$$

   and

$$T = \{\vec{v}_1, \ldots, \vec{v}_n, \vec{w}_1, \ldots, \vec{w}_j\}$$

   since $S \subseteq T$. Suppose $\vec{x} \in \operatorname{span}(S)$. Then there exist field elements $a_1, \ldots, a_n$ such that

$$\vec{x} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n.$$

   But this clearly also equals

$$\vec{x} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n + 0\vec{w}_1 + \cdots + 0\vec{w}_j.$$

   By definition $\vec{x} \in \operatorname{span}(T)$.

5. (Page 158: # 4.104(b),(c))Find the rank of each of the following matrices:

$$B = \begin{bmatrix} 1 & 2 & -3 & 2 \\ 1 & 3 & -2 & 0 \\ 3 & 8 & -7 & -2 \\ 2 & 1 & -9 & -10 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution (b) We apply the row operations $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 2R_1$:

$$B \sim \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & -3 & -3 & 6 \end{bmatrix}.$$

$R_3 \to R_3 - 2R_2$, $R_4 \to R_4 + 3R_2$

$$B \sim \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This last matrix has 2 pivots and thus the rank of B is 2.

(c) We apply the row operations $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 5R_1$, $R_4 \to R_4 + R_1$.

$$C \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 3 & -9 \\ 0 & -1 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$, $R_4 \to R_4 + R_2$

$$C \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$C \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}..$$

So the rank of the matrix is 3.

6. (Page 166: # 4.107) Determine which of the following matrices have the same row space:

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 10 \\ 3 & -5 & 1 \end{bmatrix}.$$

Solution Two matrices have the same row space if and only if they have the same non-zero rows in their reduced echelon forms.

We now row reduce the matrices beginning with A:
$R_2 \to R_2 - 3R_1$

$$A \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 8 \end{bmatrix}$$

$R_2 \to R_2/2$

$$A \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

$R_1 \to R_1 + 2R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}$$

Row reduction of B:
$R_2 \to R_2 - 2R_1$

$$B \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -5 \end{bmatrix}$$

$R_2 \to R_2/5$

$$B \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$B \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Row reduction of C: $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$C \sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 4 \\ 0 & -2 & -8 \end{bmatrix}.$$

$R_3 \to R_3 + 2R_2$

$$C \sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

$R_1 \to R_1 + R_2$

$$C \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

The reduced echelon form of A, B, and C are respectively

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

So A and C the have the same row space, and the row spaces of A and B are different.

7. (Page 159: # 4.111) Show that if any row is deleted from a matrix in echelon (repectively,row canonical) form, then the resulting matrix is still in echelon (respectively, row canonical) form.

   Solution We prove this in the second case (row canonical form).

   Let A be an $m \times n$ matrix with rows $\vec{R}_1, \ldots, \vec{R}_m$. That is

$$A = \begin{bmatrix} -\vec{R}_1- \\ -\vec{R}_2- \\ \vdots \\ -\vec{R}_m- \end{bmatrix}$$

Suppose we remove the i-th row $\vec{R}_i$ with $1 \leq i \leq m$. Note that the pivot in $\vec{R}_i$ (if there is one) is to the right of the pivot in $\vec{R}_{i-1}$ and the pivot in $\vec{R}_{i+1}$ (if there is one) is to the right of the one in $\vec{R}_i$. Therefore the pivot in $\vec{R}_{i+1}$ must be to the right of the one in $\vec{R}_{i-1}$.

Let j denote the index of one of the other non-zero rows (i.e $j \neq i$). If $j < i$ it is clear that there are there are only zeros above the pivot. If $j > i$ it is also clear that there are only zeros above the pivot. (Removing the i-th row does not change this fact.)

Thus the matrix formed from removing $\vec{R}_i$ is still in row canonical form.