Solutions to Homework 7 - Math 3410

1. (Page 159: # 4.114) Suppose U and W are two-dimensional subspaces of K^3. Show that $U \cap W \neq \{\vec{0}\}$.

   Solution. Note that $U + W \subseteq K^3$ and is a subspace. Therefore

   $$\dim(U+W) \leq \dim(K^3) = 3.$$

   Note that

   $$\dim(U+W) + \dim(U \cap W) = \dim(U) + \dim(W) = 2 + 2 = 4.$$

   Therefore

   $$\dim(U \cap W) = 4 - \dim(U+W) \geq 4 - 3 = 1.$$

   Therefore $U \cap W$ is at least a one-dimensional subspace of K^3 and thus $U \cap W \neq \{\vec{0}\}$.

2. (Page 159: # 4.115) Suppose U and W are subspaces of V such that $\dim(U) = 4$, $\dim(W) = 5$, and $\dim(V) = 7$. Find the possible dimensions of $U \cap W$.

   Solution. Observe that $U + W$ is a subspace of V and therefore

   $$\dim(U+W) \leq \dim(V) = 7.$$

   On the other hand, $U \subseteq U + W$ and $W \subseteq U + W$. Thus

   $$4 = \dim(U) \leq \dim(U+W) \text{ and } 5 = \dim(W) \leq \dim(U+W).$$

   We deduce that

   $$5 \leq \dim(U+W) \leq 7$$

   Since

   $$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U+W) = 4 + 5 - \dim(U+W) = 9 - \dim(U+W)$$

   and the possible values of $\dim(U+W)$ are $5, 6$, and 7, then possible values of $\dim(U \cap W)$ are $9 - 5 = 4, 9 - 6 = 3$, and $9 - 7 = 2$.

3. (Page 160: # 4.118) Let U_1, U_2, U_3 be the following subspaces of $\mathbb{R}^3$:

   $$U_1 = \{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a = c\} = \{\begin{pmatrix} a \\ b \\ a \end{pmatrix} \mid a \in \mathbb{R}\} = span(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix})$$

   $$U_2 = \{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a + b + c = 0\} = \{\begin{pmatrix} -b-c \\ b \\ c \end{pmatrix} \mid b, c \in \mathbb{R}\} = span(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix})$$

   $$U_3 = \{\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}\} = span(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$$

   Show that (a) $\mathbb{R}^3 = U_1 + U_2$, (b) $\mathbb{R}^3 = U_2 + U_3$, (c) $\mathbb{R}^3 = U_1 + U_3$. When is the sum direct?

   Solution. Note that $\dim(U_1) = 2, \dim(U_2) = 2$, and $\dim(U_3) = 1$. This is simple to check. In the first two cases, we just observe that the elements which span the spaces are not multiples of each other and hence linearly independent. The third case is clear since any single vector is linearly independent. Let

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(i) Note that $U_1 + U_2$ contains both U_1 and U_2. Thus it contains $\vec{u}_1, \vec{u}_2, \vec{v}_1, \vec{v}_2$ and moreover it contains the span of these four vectors. So

$$W = span(\vec{u}_1, \vec{u}_2, \vec{v}_1, \vec{v}_2) \subseteq U_1 + U_2.$$

We interpret W as the column space of A where

$$A = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

It may be shown that $A \sim U$ where

$$U = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Since U has rank 3. It follows that $\dim(\text{colsp}(A)) = \dim(W) = 3$. Therefore

$$3 = \dim(W) \leq \dim(U_1 + U_2) \leq \dim(\mathbb{R}^3) = 3.$$

Consequently, $U_1 + U_2 = \mathbb{R}^3$.

(ii) Note that $U_2 + U_3$ contains both U_2 and U_3. Thus it contains $\vec{v}_1, \vec{v}_2, \vec{w}$ and moreover it contains the span of these four vectors. So

$$W = span(\vec{v}_1, \vec{v}_2, \vec{w}) \subseteq U_2 + U_3.$$

We interpret W as the column space of A where

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

It follows that $A \sim U$ where

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since U has rank 3. It follows that $\dim(\text{colsp}(A)) = \dim(W) = 3$. Therefore

$$3 = \dim(W) \leq \dim(U_2 + U_3) \leq \dim(\mathbb{R}^3) = 3.$$

Consequently, $U_2 + U_3 = \mathbb{R}^3$.

(iii) Note that $U_1 + U_3$ contains both U_1 and U_3. Thus it contains $\vec{u}_1, \vec{u}_2, \vec{w}$ and moreover it contains the span of these four vectors. So

$$W = span(\vec{u}_1, \vec{u}_2, \vec{w}) \subseteq U_1 + U_3.$$

We interpret W as the column space of A where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

It follows that $A \sim U$ where

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since U has rank 3. It follows that $\dim(\text{colsp}(A)) = \dim(W) = 3$. Therefore

$$3 = \dim(W) \leq \dim(U_1 + U_3) \leq \dim(\mathbb{R}^3) = 3.$$

Consequently, $U_1 + U_3 = \mathbb{R}^3$.

We now determine whether each of these spaces are direct sums are not. Note that

$$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W)$$

Thus

$$\dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 + U_2) = 2 + 2 - 3 = 1$$

$$\dim(U_2 \cap U_3) = \dim(U_2) + \dim(U_3) - \dim(U_2 + U_3) = 2 + 1 - 3 = 0$$

$$\dim(U_1 \cap U_3) = \dim(U_1) + \dim(U_3) - \dim(U_1 + U_3) = 2 + 1 - 3 = 0$$

It follows that $U_1 \cap U_2 \neq \{\vec{0}\}$ since $\dim(U_1 \cap U_2) = 1$ and $U_2 \cap U_3 = \{\vec{0}\}$ and $U_1 \cap U_3 = \{\vec{0}\}$ since $\dim(U_2 \cap U_3) = \dim(U_2 \cap U_3) = 0$. Therefore the decomposition in (a) is not a direct sum but the decompositions of (b) and (c) are direct sums.