EXTREME VALUES OF $\zeta'(\rho)$

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ABSTRACT. In this article we exhibit small and large values of $\zeta'(\rho)$ by applying Soundararajan's resonance method. Our results assume the Riemann hypothesis.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta function and let ρ denote a non-trivial zero of this function. A famous conjecture due to Riemann asserts that all non-trivial zeros ρ have real part equal to one-half. This is the Riemann hypothesis. In this article we are concerned with large and small values of $\zeta'(\rho)$. Note that if $|\zeta'(\rho)|$ were small then we would expect a small gap between consecutive zeros of $\zeta(s)$ nearby. An extreme example of this phenomenon is that if ρ is a multiple zero of the zeta function then $\zeta'(\rho) = 0$. On the other hand, if $\zeta'(\rho)$ were large we would expect a large gap between zeros of $\zeta(s)$ nearby. This has been observed numerically in Odlyzko [11]. Also Soundararajan [15] has conjectured that a zero of $\zeta'(s)$ close to the half line would correspond to nearby pair of close zeros of the zeta function on the half-line. Recall that the phenomenon of a close pair of zeros of $\zeta(s)$ is referred to as Lehmer's phenomenon. One reason for our interest in such small spaces between the zeros of zeta is due to their connection to the non-existence of Landau-Siegel zeros. This connection was first noticed by Montgomery in [6] and Montgomery and Weinberger in [8]. This idea was further explored by Conrey and Iwaniec in [3]. The problem of the true size of $\zeta'(\rho)$ remains an open question. Under the Riemann hypothesis, we have by an argument of Littlewood, that there exists $c_0 > 0$ such that

$$|\zeta'(\rho)| \ll \exp\left(\frac{c_0 \log|\gamma|}{\log\log|\gamma|}\right)$$

where $\gamma = \mathrm{Im}(\rho)$. This last notation shall be employed throughout the article. On the other hand, we are also interested in small values of $|\zeta'(\rho)|$. Consider $\Theta = \inf\{\ c \mid |\zeta'(\rho)|^{-1} \ll \gamma^c\ \}$ defined by Gonek [4] in his study of $M(x)$, the summatory function of the Möbius function. Since the Riemann hypothesis implies $|\zeta'(\rho)| \ll |\rho|^{\epsilon}$ one expects that $\Theta \geq 0$. On the other hand, the GUE conjecture which asserts that the that distribution of consecutive zeros of the zeta function obey the GUE distribution suggests that $\Theta = \frac{1}{3}$ and hence we should have

$$|\zeta'(\rho)| \ll \gamma^{-1/3+\epsilon}$$

infinitely often.

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In this article we shall produce results exhibiting both large and small values of $|\zeta'(\rho)|$. These results are obtained by a novel idea due to Soundararajan [16]. The method, coined the resonance method, will be explained shortly. We begin with the large values result.

Theorem 1. *Assume the Riemann hypothesis. For each $A > 0$ we have*

$$|\zeta'(\rho)| \gg_A (\log|\gamma|)^A$$

for infinitely many γ.

I would like to note that Soundararajan has informed me that he has proven that

$$\sum_{0<\gamma<T} |\zeta'(\rho)|^{2k} \gg_k T(\log T)^{(k+1)^2} \tag{1}$$

by the lower bound method of Rudnick and Soundararajan [12], [13]. Clearly, (1) implies Theorem 1. However, as this remains unpublished, we present our proof of Theorem 1. Thus under the Riemann hypothesis, the lower bound method [12], [13] can give omega results for $\zeta'(\rho)$ of the same strength as the resonance method [16]. This stems from the fact that we are unable to evaluate a certain weighted sum of $\zeta'(\rho)$ without making assumptions about the zeros of Dirichlet L-functions (see Proposition 4 parts (*ii*) and (*iii*) that follow). If we are willing to assume an additional hypothesis concerning the location of the zeros of Dirichlet L-functions we can improve Theorem 1 significantly and we can obtain a result of the same quality as Soundararajan's results [16]. We shall require the following:

Large zero-free region conjecture. There exists a positive constant c_0' sufficiently large such that for each $q \geq 1$ and each character χ modulo q the Dirichlet L-function $L(s,\chi)$ does not vanish in the region

$$\sigma \geq 1 - \frac{c_0'}{\log\log(q(|t|+4))}$$

where $s = \sigma + it$.

Note that this conjecture is significantly weaker than the generalized Riemann hypothesis. However, it is a sufficiently strong hypothesis to rule out the existence of Siegel zeros. Recall that the classical zero-free region for Dirichlet L-functions is $L(s,\chi)$ does not vanish in the region

$$\sigma \geq 1 - \frac{c_1}{\log(q(|t|+3))}$$

for some $c_1 > 0$ with the possible exception of one simple real zero in the case χ is quadratic.

Theorem 2. *Assume the Riemann hypothesis and the large zero-free region conjecture. There are arbitrarily large values of γ such that*

$$|\zeta'(\rho)| \gg \exp\left(c_2 \sqrt{\frac{\log|\gamma|}{\log\log|\gamma|}} \right)$$

where $c_2 = \frac{1}{\sqrt{2}} - \epsilon$ is valid.

We also prove a result for small values of $|\zeta'(\rho)|$. Surprisingly, this proof is significantly easier than the proof of Theorem 2.

Theorem 3. *Assume the Riemann hypothesis. We have*

$$|\zeta'(\rho)| \ll \exp\left(-c_3\sqrt{\frac{\log|\gamma|}{\log\log|\gamma|}}\right)$$

for infinitely many γ where $c_3 = \sqrt{\frac{2}{3}} - \epsilon$ is valid.

2. Notation

We shall use Vinogradov's notation $f(x) \ll g(x)$ to mean there exists a $C > 0$ such that $|f(x)| \leq Cg(x)$ for all x sufficiently large. We denote $f(x) = O(g(x))$ to mean the same thing. Also, $f(x) = o(g(x))$ means $f(x)/g(x) \to 0$ as $x \to \infty$. We shall consider arbitrary sequences x_n supported on an interval $[1, M]$ and we employ the notation

$$||x_n||_\infty = \max_{n \leq M} |x_n| \text{ and } ||x_n||_p = (\sum_{n \leq M} |x_n|^p)^{1/p} .$$

We now define some basic arithmetic functions. We define $\mu(n)$, the Mobius function, to be the coefficient of n^{-s} in the Dirichlet series $\zeta(s)^{-1} = \sum_{n=1}^{\infty} \mu(n) n^{-s}$. We define $\Lambda_k(n)$ to be the coefficient of n^{-s} in the Dirichlet series of $(-1)^k \zeta^{(k)}(s)/\zeta(s)$. Another way to express this is $\Lambda_k(n) = (\mu * \log^k)(n)$. Note that $\Lambda_k(n)$ is supported on those integers with at most k distinct prime factors. We define $\tau_k(n)$, the k-the divisor function, to be the coefficient of n^{-s} in the Dirichlet series $\zeta(s)^k = \sum_{n=1}^{\infty} \tau_k(n) n^{-s}$.

3. Explanation of the resonance method

In this section we outline the resonance method. Soundararajan [16] recently invented this simple method to find large values of $|\zeta(1/2 + it)|$ (and also other L-functions and character sums). Under the Riemann hypothesis it is known that

$$|\zeta(1/2 + it)| \ll \exp\left(\frac{c_1' \log|t|}{\log\log|t|}\right)$$

where $c_1' > 0$ is explicitly given. However, it has been proven by Montgomery [7], assuming the Riemann hypothesis, that there exist arbitrarily large t such that

$$|\zeta(1/2 + it)| \gg \exp\left(c_2'\sqrt{\frac{\log|t|}{\log\log|t|}}\right)$$

for some positive constant c_2'. Later, Balasubramanian and Ramachandra [1] gave an unconditional proof of this result with an explicit value $c_2' < 1$. The new method permits the choice $c_2' = 1 - \epsilon$. We now sketch the method. Consider the mean values

$$\int_T^{2T} \zeta(1/2 + it)|A(it)|^2\, dt \text{ and } \int_T^{2T} |A(it)|^2\, dt$$

where $A(s) = \sum_{n \leq M} x_n n^{-s}$ is a Dirichlet polynomial with arbitrary positive coefficients x_n and $y \leq T^{1-\epsilon}$. A standard calculation shows that

$$\frac{\int_T^{2T} \zeta(1/2 + it)|A(it)|^2\, dt}{\int_T^{2T} |A(it)|^2\, dt} = \left(\frac{\sum_{nu \leq M} \frac{x_n x_{nu}}{\sqrt{u}}}{\sum_{n \leq M} x_n^2}\right)(1 + o(1)) .$$

By taking absolute values we deduce that

$$\max_{T\le t\le 2T}|\zeta(1/2+it)|\ge\left(\frac{\sum_{nu\le M}\frac{x_nx_{nu}}{\sqrt{u}}}{\sum_{n\le M}x_n^2}\right)(1+o(1))\ . \tag{2}$$

The problem is thus reduced to optimizing the fraction on the right. Soundararajan [16] shows that the maximum of the above quotient is

$$\exp\left(\sqrt{\tfrac{\log M}{\log\log M}}(1+o(1))\right)$$

and this is obtained by choosing $x_n = f(n)$ where $f(n)$ is multiplicative and supported on squarefree numbers. We define f on the primes as follows: let $L=\sqrt{\log M\log\log M}$ and set

$$f(p)=\begin{cases}\frac{L}{\sqrt{p}\log p} & \text{if } L^2\le p\le\exp((\log L)^2)\\ 0 & \text{else}\end{cases}\ .$$

The strategy of this article is to follow the above argument. We require asymptotic formulae for the mean values

$$S_1=\sum_{0<\gamma<T}\zeta'(\rho)A(\rho)A(1-\rho)\text{ and }S_2=\sum_{0<\gamma<T}A(\rho)A(1-\rho)$$

where $A(s)=\sum_{n\le M}x_nn^{-s}$ has arbitrary real coefficients x_n, $y=T^\theta$, and $\theta<1/2$. Observe that if the Riemann Hypothesis is true then $|A(\rho)|^2=A(\rho)A(1-\rho)$ and thus

$$S_1=\sum_{0<\gamma<T}\zeta'(\rho)|A(\rho)|^2\text{ and }S_2=\sum_{0<\gamma<T}|A(\rho)|^2\ . \tag{3}$$

In fact, we shall show that S_1/S_2 is essentially the same quotient of quadratic forms as in (2).

We have the following formulae for S_1 and S_2:

Proposition 4. *(i) Suppose that $|x_n|\ll T^\epsilon$ and $\theta<1$. Then we have*

$$S_2=N(T)\sum_{m\le M}\frac{x_m^2}{m}-\frac{T}{\pi}\sum_{m\le M}\frac{(\Lambda*x)(m)x_m}{m}+o(T) \tag{4}$$

where $N(T)$ is the number of zeros of the zeta function in the box $0\le\mathrm{Re}(s)\le 1$, $0\le\mathrm{Im}(s)\le T$.
(ii) Suppose that $|x_n|\ll\tau_r(n)(\log T)^C$ for some $C>0$ and $\theta<1/2$. Then we have

$$S_1=\frac{T}{2\pi}\left(\sum_{nu\le M}\frac{x_ux_{nu}r_0(n)}{nu}+\sum_{\substack{a,b\le M\\(a,b)=1}}\frac{r_1(a,b)}{ab}\sum_{g\le\min(\frac{M}{a},\frac{M}{v})}\frac{x_{ag}x_{bg}}{g}\right)+o(T) \tag{5}$$

where

$$\begin{aligned}r_0(n)&=\tfrac12P_2(\log(\tfrac{T}{2\pi}))-P_1(\log(\tfrac{T}{2\pi}))\log n-\tfrac12(\log n)^2+(\Lambda*\log)(n)\ ,\\ r_1(a,b)&=\tfrac12\Lambda_2(a)-R_1\left(\log\left(\tfrac{T}{b}\right)\right)\Lambda(a)-\tilde{R}_1\left(\log\left(\tfrac{T}{b}\right)\right)\alpha_1(a)-\alpha_2(a)\ ,\end{aligned}$$

$P_2,P_1,R_1,\tilde{R}_1$ are monic polynomials of degrees 2,1,1,1 respectively. α_2,α_1 are arithmetic functions. α_2 is supported on a with $\omega(a)\le 2$ and α_1 is supported on prime powers. Moreover, $\alpha_1(p^j)\ll\frac{\log p}{p}$, $\alpha_2(p^j)\ll\frac{j(\log p)^2}{p}$, and $\alpha_2(p^jq^k)\ll(\log p)(\log q)(p^{-1}+q^{-1})$.

(iii) Assume the large zero-free region conjecture. The formula for S_1 in (ii) remains valid under the assumption that $x_n = \sqrt{n} f(n)$ and $\theta < 1/3$.

Proof. The proofs of (*ii*) and (*iii*) may be found in Theorem 1.3 of [10]. The formula for S_2 in (*i*) is mentioned without proof on page 6 of [2]. It can be proven by following the argument of [9] Lemma 3. $\square$

From Proposition 4, we can explain our strategy for proving Theorem 2. We shall show that in the formulae (5) and (4) for S_1 and S_2 the significant terms are

$$\frac{T\log^2(\frac{T}{2\pi})}{4\pi}\sum_{nu\le M}\frac{x_u x_{nu}}{nu} \text{ and } \frac{T\log(\frac{T}{2\pi})}{2\pi}\sum_{m\le M}\frac{x_m^2}{m}$$

respectively. By choosing $x_n = \sqrt{n} f(n)$ we see that

$$\max_{T\le\gamma\le 2T}|\zeta'(\rho)| \ge \frac{S_1}{S_2} \approx \frac{\log(\frac{T}{2\pi})}{2}\left(\frac{\sum_{rn\le M}\frac{f(n)f(nr)}{\sqrt{r}}}{\sum_{n\le M} f(n)^2}\right) = \exp\left(\sqrt{\frac{\log M}{\log\log M}}(1+o(1))\right) .$$

This is the essential content of Theorem 2. In order to make this argument rigorous, we will show that each of the other terms in the formulae for S_1 and S_2 are smaller than the principal terms. The argument for Theorem 3 is very similar. In this case we consider

$$S_3 = \sum_{T<\gamma<2T}\zeta'(\rho)^{-1}|A(\rho)|^2 \text{ and } S_2 = \sum_{T<\gamma<2T}|A(\rho)|^2 .$$

As before we will show that the ratio S_3/S_2 gives rise to the same quadratic form as in (2).

4. Large values of $\zeta'(\rho)$: Proof of Theorem 1

In this section we prove Theorem 1. As explained previously our strategy is to evaluate asymptotically S_1/S_2 for a certain choice of coefficients. As we are only assuming the Riemann hypothesis, we are restricted to choosing $x_n = \tau_r(n)$ with $r \in \mathbb{N}$. In the course of this calculation, we shall encounter several other multiplicative functions. We define

$$f_1(n) = \prod_{p^e||n}\frac{\sum_{j=0}^{\infty}\frac{\tau_r(p^{e+j})}{p^j}}{\sum_{j=0}^{\infty}\frac{\tau_r(p^j)}{p^j}} \text{ and } f_2(n) = \prod_{p^e||n}\frac{\sum_{j=0}^{\infty}\frac{\tau_r(p^{e+j})\tau_r(p^j)}{p^j}}{\sum_{j=0}^{\infty}\frac{\tau_r(p^j)^2}{p^j}} .$$

Note that for $i = 1, 2$ $f_i(p) = r(1 + O(p^{-1}))$. The asymptotic evaluation of S_1 will require the evaluation of several sums of standard arithmetic functions. We shall employ the following:

Lemma 5. *Let $a, b, k, r, u \in \mathbb{N}$.*
(i)

$$\sum_{n\le x}\tau_r(nu) = \frac{f_1(u)x(\log x)^{r-1}}{(r-1)!}(1 + O((\log x)^{-1})) .$$

(ii)

$$\sum_{n\le x}\tau_r(n)f_1(n) = \frac{C_0 x(\log x)^{r^2-1}}{(r^2-1)!}(1 + O((\log x)^{-1}))$$

where

$$C_0=\prod_p\left(1-\frac{1}{p}\right)^r\sum_{j=0}^{\infty}\frac{\tau_r(p^j)f_1(p^j)}{p^j}=\prod_p\left(1-\frac{1}{p}\right)^{r^2+r}\sum_{j=0}^{\infty}\frac{\tau_r(p^j)\tau_{r+1}(p^j)}{p^j}\ . \quad (6)$$

(iii)

$$\sum_{n\le x}f_2(n)=\frac{C_1x(\log x)^{r-1}}{(r-1)!}(1+O((\log x)^{-1})$$

where

$$C_1=\prod_p(1-1/p)^r\sum_{j=0}^{\infty}\frac{f_2(p^j)}{p^j}=\prod_p(1-1/p)^r\,\frac{\sum_{j=0}^{\infty}\frac{\tau_r(p^j)\tau_{r+1}(p^j)}{p^j}}{\sum_{j=0}^{\infty}\frac{\tau_r(p^j)^2}{p^j}}\ . \quad (7)$$

(iv)

$$\sum_{n\le x}\frac{\tau_r(an)\tau_r(bn)}{n}=C_2f_2(a)f_2(b)\frac{(\log x)^{r^2}}{(r^2)!}(1+O((\log x)^{-1}))$$

where

$$C_2=\prod_p(1-1/p)^{r^2}\sum_{j=0}^{\infty}\frac{\tau_r(p^k)^2}{p^k}\ .$$

Notice that it follows immediately from (6) and (7) that $C_0=C_1C_2$.

(v)

$$\sum_{n\le x}\Lambda_k(n)=kx(\log x)^{k-1}(1+O((\log x)^{-1}))\ .$$

(vi) For $i=1,2$

$$\sum_{n\le x}\Lambda(n)f_i(n)=rx(1+O((\log x)^{-1}))\ .$$

(vii) For $i=1,2$

$$\sum_{n\le x}\Lambda_2(n)f_i(n)=(r^2+r)x(\log x)(1+O((\log x)^{-1}))\ .$$

Proof. Since the proofs of $(i)-(iv)$ are very similar we shall just prove part (iv). We give a sketch of the proof as the argument is standard (see for example [14]). We define the Dirichlet series $H(s)=\sum_{n=1}^{\infty}\tau_r(an)\tau_r(bn)n^{-s}$ and since τ_r is multiplicative we have the factorization

$$H(s)=\prod_{(p,ab)=1}\left(\sum_{k=0}^{\infty}\frac{\tau_r(p^k)^2}{p^{ks}}\right)\prod_{p^e||a}\sum_{k=0}^{\infty}\frac{\tau_r(p^{e+k})\tau_r(p^k)}{p^{ks}}\prod_{p^f||b}\sum_{k=0}^{\infty}\frac{\tau_r(p^k)\tau_r(p^{f+k})}{p^{ks}}\ .$$

Next we define for $s\in\mathbb{C}$ and $n\in\mathbb{N}$

$$F(s;n)=\prod_{p^e||n}\left(\frac{\sum_{k=0}^{\infty}\frac{\tau_r(p^{e+k})\tau_r(p^k)}{p^{ks}}}{\sum_{k=0}^{\infty}\frac{\tau_r(p^k)^2}{p^{ks}}}\right)\ ,\ G(s)=\prod_p(1-1/p^s)^{r^2}\sum_{k=0}^{\infty}\frac{\tau_r(p^k)^2}{p^{ks}}$$

and thus $H(s)=\zeta(s)^{r^2}F(s,ab)G(s)$. Moreover, we notice that $F(1;n)=f_2(n)$ and $G(1)=C_2$. By Perron's formula,

$$\sum_{n\le x}\frac{\tau_r(an)\tau_r(bn)}{n}=\frac{1}{2\pi i}\int_{\kappa-iU}^{\kappa+iU}H(s+1)\,\frac{x^sds}{s}+O\left(\frac{(\log x)^{r^2}}{U}+\frac{1}{x^{1-\epsilon}}\left(1+x\frac{\log U}{U}\right)\right)$$

with $\kappa = (\log x)^{-1}$. Let $\Gamma(U)$ denote the contour consisting of $s \in \mathbb{C}$ such that $\mathrm{Re}(s) = -\frac{c'}{\log(|\mathrm{Im}(s)|+2)}$ and $|\mathrm{Im}(s)| \le U$ for an appropriate $c' > 0$. We deform the contour past $\mathrm{Re}(s) = 0$ line to $\Gamma(U)$ picking up the residue at $s = 0$. The residue at $s = 0$ equals

$$C_2 f_2(a) f_2(b) \frac{(\log x)^{r^2}}{(r^2)!}(1 + O((\log x)^{-1}))$$

which corresponds to the main term. Employing standard bounds for $\zeta(s)$ in the zero-free region we can show that contribution of the integral on $\Gamma(U)$ is smaller than the main term for an appropriate choice of U by at least one factor of $\log x$. Part (v) is a well known fact. Part (vi) follows from the fact that Λ is supported on the prime powers and $\Lambda(p^j) = \log(p)$. Part (vii) follows from the fact that Λ_2 is supported on those n with $\omega(n) \le 2$ and moreover $\Lambda_2(pq) = 2\log p \log q$, $\Lambda_2(p) = (\log p)^2$, and $f_i(p) = r(1 + O(p^{-1}))$. $\square$

We are now prepared to prove Theorem 1. In the course of the proof, we will encounter the following integrals:

$$\begin{aligned} i(u,v) &:= \int_0^1 x^u (1-x)^v \, dx = \frac{u!v!}{(u+v+1)!} \ , \\ c_X(u,v) &:= \int_1^X \frac{(\log X/t)^u (\log t)^v}{t} \, dt = (\log X)^{u+v+1} i(u,v) \end{aligned} \tag{8}$$

where $X \ge 1$.

Proof of Theorem 1. By Proposition 4 we may write $S_1 = \tilde{S}_1 + o(T)$ where

$$\tilde{S}_1 = \frac{T}{2\pi}\left(\sum_{nu \le M} \frac{x_u x_{nu} r_0(n)}{nu} + \sum_{\substack{a,b \le M \\ (a,b)=1}} \frac{r_1(a,b)}{ab} \sum_{g \le \min(\frac{M}{a}, \frac{M}{v})} \frac{x_{ag} x_{bg}}{g} \right)$$

and

$$\begin{aligned} r_0(n) &= \frac{1}{2} P_2(\log(\tfrac{T}{2\pi})) + \sum_{d|n} g(d) \ , \\ g(d) &= -(P_1(\log(\tfrac{T}{2\pi})) + \log d)\Lambda(d) + \frac{\Lambda_2(d)}{2} \ . \end{aligned} \tag{9}$$

Thus we have $\tilde{S}_1 = \frac{T}{2\pi}\left(\frac{P_2(\log(T/2\pi))}{2} T_1 + T_2 + T_3 \right)$ where

$$\begin{aligned} T_1 &= \sum_{nu \le M} \frac{x_u x_{nu}}{nu} \ , \\ T_2 &= \sum_{dnu \le M} \frac{g(d) x_u x_{dnu}}{dnu} \ , \\ T_3 &= \sum_{\substack{a,b \le M \\ (a,b)=1}} \frac{r_1(a,b)}{ab} \sum_{g \le \min(\frac{M}{a}, \frac{M}{v})} \frac{x_{ag} x_{bg}}{g} \ . \end{aligned} \tag{10}$$

4.1. **Evaluation of T_1.** Now by Lemma 5 (i) and (ii) we have

$$\begin{aligned}T_1 &= \sum_{u\le M}\frac{\tau_r(u)}{u}\int_{1^-}^{M/u} t^{-1} d\left(\sum_{n\le t}\tau_r(nu)\right)\\ &\sim \sum_{u\le M}\frac{\tau_r(u)f_1(u)}{u}\frac{(\log M/u)^r}{r!} = \frac{1}{r!}\int_{1^-}^{M}\log(M/t)^r t^{-1} d\left(\sum_{u\le t}\tau_r(u)f_1(u)\right)\\ &\sim \frac{1}{r!}\int_1^M \frac{(\log(M/t)^r}{t}\frac{C_0(\log t)^{r^2-1}}{(r^2-1)!}\,dt\ .\end{aligned}$$

By (8) it follows that

$$T_1 \sim \frac{C_0}{r!(r^2-1)!}c_M(r,r^2-1) = \frac{C_0(\log M)^{r^2+r}}{(r^2+r)!}\ .$$

4.2. **Evaluation of T_2.** Since the calculation of T_2 and T_3 are rather similar to that of T_1 we shall not record every step of their calculation. By Lemma 5 (i) we have

$$T_2 \sim \sum_{d\le M}\frac{g(d)}{d}\sum_{u\le M/d}\frac{\tau_r(u)}{u}\frac{f_1(du)\log(M/du)^r}{r!}\ .$$

As g is supported on those integers d with $\omega(d)\le 2$ we have

$$\begin{aligned}T_2 &\sim \sum_{d\le M}\frac{g(d)f_1(d)}{d}\sum_{u\le M/d}\frac{\tau_r(u)}{u}\frac{f_1(u)\log(M/du)^r}{r!}\\ &= \frac{1}{r!}\sum_{d\le M}\frac{g(d)f_1(d)}{d}\int_1^{M/d}\frac{\log(M/dt)^r}{t}\frac{C_0(\log t)^{r^2-1}}{(r^2-1)!}\,dt\\ &= \frac{C_0}{(r^2+r)!}\sum_{d\le M}\frac{g(d)f_1(d)}{d}(\log M/d)^{r^2+r}\end{aligned}$$

where we have invoked Lemma 5 (ii) and (8). By (9), Lemma 5 (vi) and (vii) we obtain

$$\sum_{n\le x} g(n)f_1(n) \sim x\left(\frac{r^2-r}{2}\log x - rP_1(\log(\tfrac{T}{2\pi}))\right)\ .$$

From this we deduce

$$\begin{aligned}T_2 &= \frac{C_0}{(r^2+r)!}\int_1^M\frac{(\log M/t)^{r^2+r}}{t}\left(\frac{r^2-r}{2}\log(t) - rP_1(\log(\tfrac{T}{2\pi}))\right)dt\\ &\sim \frac{C_0}{(r^2+r)!}\left(\frac{r^2-r}{2}c_M(r^2+r,1) - \frac{r}{\theta}c_M(r^2+r,0)\right)(\log M)^{r^2+r+2}\end{aligned}$$

and it follows from (8) that

$$T_2 \sim \frac{C_0(\log M)^{r^2+r+2}}{(r^2+r+2)!}\left(\frac{r^2-r}{2} - \frac{r}{\theta}(r^2+r+2)\right)\ .$$

4.3. **Evaluation of T_3.** By Lemma 5 (*iv*) it follows that

$$T_3 \sim \frac{C_2}{(r^2)!} \sum_{\substack{a,b \le M \\ (a,b)=1}} \frac{r_1(a,b) f_2(a) f_2(b)}{ab} \left(\log \min \left(\frac{M}{a}, \frac{M}{b} \right) \right)^{r^2}$$

where $r_1(a,b)$ is defined by (6). We shall write this last sum as $T_3' + T_3''$ where T_3' is the sum over the terms for which $a < b \le M$ and T_3'' consists of the terms for which $b < a \le M$. We have

$$T_3' \sim \frac{C_2}{(r^2)!} \sum_{b \le M} \frac{f_2(b) \log(M/b)^{r^2}}{b} \sum_{\substack{a<b \\ (a,b)=1}} \frac{(1/2)\Lambda_2(a) - \Lambda(a) R_1(\log(T/b))}{a} f_1(a)$$

since it may be checked that the contribution from the term $-\tilde{R}_1(\log(T/b))\,\alpha_1(a) - \alpha_2(a)$ is $\ll (\log T)^{r^2+r+1}$. By Lemma 5 (*vi*) and (*vii*)

$$\sum_{a \le x} f_1(a) \left((1/2)\Lambda_2(a) - \Lambda(a) R_1(\log(T/b)) \right) \sim \frac{(r^2+r)}{2} x \log x - r R_1(\log(T/b)) x$$

and it follows that

$$\begin{aligned} T_3' &\sim \frac{C_2}{(r^2)!} \sum_{b \le M} \frac{f_2(b) \log(M/b)^{r^2}}{b} \int_1^b \frac{(1/2)(r^2+r)\log t - r R_1(\log(T/b))}{t} \, dt \\ &= \frac{C_2}{(r^2)!} \sum_{b \le M} \frac{f_2(b) \log(M/b)^{r^2}}{b} \left(\frac{r^2+r}{4} (\log b)^2 - r R_1(\log(T/b)) \log b \right) . \end{aligned}$$

By Lemma 5 (*iii*)

$$\begin{aligned} T_3' &= \frac{C_2}{(r^2)!} \int_1^M \frac{\log(M/t)^{r^2}}{t} \left(\frac{r^2+r}{4} (\log t)^2 - r R_1(\log(T/t)) \log t \right) \frac{C_1}{(r-1)!} (\log t)^{r-1} \, dt \\ &= \frac{C_0 (\log M)^{r^2+r+2}}{(r^2+r+2)!} \left(\frac{(r^2+5r)(r+1)r}{4} - \frac{r^2(r^2+r+2)}{\theta} \right) . \end{aligned}$$

Next, we consider those terms with $b < a \le M$. We have

$$T_3'' \sim \frac{C_2}{(r^2)!} \sum_{a \le M} \sum_{\substack{b<a \\ (a,b)=1}} \frac{(1/2)\Lambda_2(a) - \Lambda(a) R_1(\log(T/b))}{a} \frac{f_1(a) f_2(b) \log(M/a)^{r^2}}{b}$$

since we can show, as before, that the contribution from the term $-\tilde{R}_1(\log(T/b))\,\alpha_1(a) - \alpha_2(a)$ is $\ll (\log T)^{r^2+r+1}$. Since $\sum_{b \le x} f_2(b) \sim \frac{C_1}{(r-1)!} x (\log x)^{r-1}$, a similar calculation as above yields

$$T_3'' \sim \frac{C_0}{(r^2)! r!} \int_{1^-}^M \frac{\log(M/t)^{r^2} (\log t)^r}{t} d\sigma(t)$$

with $\sigma(t)=\sum_{a\le t}\left((\Lambda_2(a)/2-\Lambda(a)\log T)+\frac{r}{r+1}\Lambda(a)\log(a)\right)f_2(a)$. By Lemma 5 (vi) and (vii) $\sigma(t)\sim\left(\frac{r^2+r}{2}+\frac{r^2}{r+1}\right)t\log t-rt(\log T)$ and thus

$$\begin{aligned}T_3''&\sim\frac{C_0}{(r^2)!r!}\left(\left(\frac{r^2+r}{2}+\frac{r^2}{r+1}\right)c_M(r^2,r+1)-r(\log T)c_M(r^2,r)\right)\\&=\frac{C_0(\log M)^{r^2+r+2}}{(r^2+r+2)!}\left(\left(\frac{(r^2+r)(r+1)}{2}+r^2\right)-\frac{r(r^2+r+2)}{\theta}\right)\ .\end{aligned}$$

Collecting our results for T_1,T_2, and $T_3=T_3'+T_3''$ we have

$$\begin{aligned}S_1&\sim\frac{C_0T(\log M)^{r^2+r+2}}{(r^2+r+2)!}\left(\frac{(r^2+r+2)(r^2+r+1)}{\theta^2}+\left(\frac{r(r-1)}{2}-\frac{r(r^2+r+2)}{\theta}\right)\right.\\&\quad\left.+(r^2+r)\left(\frac{r^2+5r}{4}+\frac{r+1}{2}+\frac{r^2}{r^2+r}\right)-\frac{(r^2+r+2)(r^2+r)}{\theta}\right)\\&\ge\frac{C_0T(\log M)^{r^2+r+2}}{(r^2+r+2)!}\frac{r^2+r+2}{\theta^2}(r^2+r+1-\theta(r^2+2r))\\&\gg\frac{r^4T(\log M)^{r^2+r+2}}{\theta^2(r^2+r+2)!}\end{aligned}$$

for $0<\theta<\frac{1}{2}$ and $r\in\mathbb{N}$. On the other hand, we have the simple bound

$$S_2\le\frac{T\log(\frac{T}{2\pi})}{2\pi}\sum_{m\le M}\frac{\tau_r(m)^2}{m}\ll\frac{T}{\theta}(\log M)^{r^2+1}$$

and thus $\max_{T\le\gamma\le 2T}|\zeta'(\rho)|\ge\left|\frac{S_1}{S_2}\right|\gg_r(\log M)^{r+1}\gg(\log T)^{r+1}$. $\square$

5. Larger values of $\zeta'(\rho)$: Proof of Theorem 2

In this section we shall evaluate S_1/S_2 for the choice $x_n=\sqrt{n}f(n)$. Before embarking on this task we will require a few results concerning the coefficients $f(n)$. Moreover, we shall encounter several other multiplicative functions. We define g and h to be multiplicative functions supported on the squarefree numbers. Their values at any prime p are given by

$$g(p)=1+f(p)^2 \text{ and } h(p)=1+f(p)p^{-1/2}\ .$$

It will also be convenient to introduce the notation

$$\mathcal{Q}_1=\prod_p\left(1+f(p)^2+\frac{f(p)}{\sqrt{p}}\right)\ ,\ \mathcal{Q}_2=\prod_p\left(1+f(p)^2\right)\ .$$

Lemma 6. (i)

$$\sum_{nu\le M}\frac{f(u)f(nu)}{\sqrt{n}}=\mathcal{Q}_1(1+o(1))\ ,$$

(ii)

$$\sum_{n\le M}f(n)^2\le\mathcal{Q}_2\ ,$$

(iii)

$$\frac{\mathcal{Q}_1}{\mathcal{Q}_2}=\exp\left(\sqrt{\frac{\log M}{\log\log M}}(1+o(1))\right)\ .$$

(iv) For $i = 1, 2$

$$\sum_{a \le M} \frac{\Lambda_i(a) f(a)}{\sqrt{a} g(a)} \ll (\log T)^{i/2+\epsilon} .$$

Proof. *(i)* We denote the sum to be estimated $\mathcal{S}$. Thus

$$\mathcal{S} = \sum_{n \le M} \frac{f(n)}{\sqrt{n}} \sum_{\substack{u \le M/r \\ (n,u)=1}} f(u)^2 = \sum_{n \le M} \frac{f(n)}{\sqrt{n}} \left(\prod_{(p,n)=1} (1 + f(p)^2) - \sum_{\substack{u > M/n \\ (n,u)=1}} f(u)^2 \right) .$$

By Rankin's trick the error term is bounded by

$$\sum_{n \le M} \frac{f(n)}{\sqrt{n}} \left(\frac{n}{M}\right)^{\alpha} \sum_{\substack{u=1 \\ (u,n)=1}}^{\infty} f(u)^2 u^{\alpha} \le \frac{1}{M^{\alpha}} \prod_p \left(1 + p^{\alpha} f(p)^2 + f(p) p^{\alpha - 1/2}\right)$$

for any $\alpha > 0$. On the other hand, since f is multiplicative the main term equals

$$\prod_p \left(1 + f(p)^2 + \frac{f(p)}{\sqrt{p}}\right) + O\left(\frac{1}{M^{\alpha}} \prod_p \left(1 + f(p)^2 + \frac{f(p) p^{\alpha}}{\sqrt{p}}\right)\right) .$$

We deduce

$$\mathcal{S} = \mathcal{Q}_1 + O\left(\frac{1}{M^{\alpha}} \prod_p \left(1 + p^{\alpha} f(p)^2 + \frac{f(p) p^{\alpha}}{\sqrt{p}}\right)\right) . \tag{11}$$

However, it is shown in [16] that the ratio of the error term to the main term in (11) is $\ll \exp(-\alpha \frac{\log M}{\log\log M})$ for the choice $\alpha = \frac{1}{(\log L)^3}$. It follows that $\mathcal{S} = \mathcal{Q}_1(1 + o(1))$.
(ii) We have the simple identity

$$\sum_{n \le M} f(n)^2 \le \sum_{n \ge 1} f(n)^2 = \mathcal{Q}_2 .$$

(iii) Note that

$$\frac{\mathcal{Q}_1}{\mathcal{Q}_2} = \prod_p \left(1 + \frac{f(p)}{\sqrt{p}(1 + f(p)^2)}\right) .$$

Taking logarithms of the product we see that

$$\log(\mathcal{Q}_1/\mathcal{Q}_2) = \sum_p \log\left(1 + \frac{f(p)}{\sqrt{p}(1 + f(p)^2)}\right) = \sum_{L^2 \le p \le \exp((\log L)^2)} \frac{L}{p \log p (1 + o(1))}$$

$$= \frac{L}{\log L^2}(1 + o(1)) = \sqrt{\frac{\log M}{\log\log M}}(1 + o(1)) .$$

(iv) We have

$$\sum_{a \le M} \frac{\Lambda(a) f(a)}{\sqrt{a} g(a)} = L \sum_{p \le M} \frac{1}{p g(p)} \ll L \sum_{p \le M} \frac{1}{p} \ll (\log T)^{1/2+\epsilon} .$$

Note that Λ_2 is supported on integers a satisfying $\omega(a) \le 2$ and f is supported on squarefree integers. Moreover $\Lambda_2(p) = (\log p)^2$ and $\Lambda_2(pq) = 2 \log p \log q$. From this, we deduce that

$$\sum_{a\le M}\frac{\Lambda_2(a)f(a)}{\sqrt{a}g(a)} \ll \sum_{p\le M}\frac{\Lambda_2(p)f(p)}{\sqrt{p}g(p)} + \sum_{pq\le M, p\ne q}\frac{\Lambda_2(pq)f(pq)}{\sqrt{pq}g(pq)}$$

$$\ll L\sum_{p\le M}\frac{\log p}{p} + L^2\left(\sum_{p\le M}\frac{1}{p}\right)^2 \ll (\log T)^{1+\epsilon} .$$

□

Proof of Theorem 2. We have from Proposition 4 that

$$\frac{S_1}{S_2} = \frac{\frac{1}{2}P_2(\log(\frac{T}{2\pi}))\Sigma_0 - P_1(\log(\frac{T}{2\pi}))\Sigma_1 - \frac{1}{2}\Sigma_2 + \Sigma_3 + \Sigma_4}{\log(\frac{T}{2\pi})\Sigma_5 - 2\Sigma_6} + o(1)$$

where for $i = 0, 1, 2$

$$\Sigma_i = \sum_{nu\le M}\frac{f(u)f(nu)(\log n)^i}{\sqrt{n}}$$

and

$$\Sigma_3 = \sum_{nu\le M}\frac{f(u)f(nu)(\Lambda * \log)(n)}{\sqrt{n}} ,$$

$$\Sigma_4 = \sum_{\substack{a,b\le M\\(a,b)=1}}\frac{r_1(a,b)}{\sqrt{ab}}\sum_{g\le\min(\frac{M}{a},\frac{M}{b})} f(ag)f(bg) ,$$

$$\Sigma_5 = \sum_{m\le M} f(m)^2 ,$$

$$\Sigma_6 = \sum_{mn\le M}\frac{\Lambda(n)f(m)f(mn)}{\sqrt{n}} .$$

By Lemma 6

$$\Sigma_0 = \mathcal{Q}_1(1+o(1)) \text{ and } \Sigma_5 \le \mathcal{Q}_2(1+o(1)) . \tag{12}$$

We shall prove the following bounds for the other five sums:

Lemma 7. *We have:*

$$\Sigma_1 \ll \mathcal{Q}_1(\log T)^{1/2+\epsilon} ,\ \Sigma_2, \Sigma_3 \ll \mathcal{Q}_1(\log T)^{1+\epsilon} ,$$
$$\Sigma_4 \ll \mathcal{Q}_1(\log T)^{3/2+\epsilon} ,\ \Sigma_6 \ll \mathcal{Q}_2(\log T)^{1/2+\epsilon} .$$

Theorem 2 now easily follows. We deduce from (12) and Lemma 7 that

$$\mathcal{S}_1 = (1/2)\mathcal{Q}_1\log^2(\tfrac{T}{2\pi})\left(1 + O((\log T)^{-1/2+\epsilon})\right)$$

and $\mathcal{S}_2 \le \mathcal{Q}_2\log(\frac{T}{2\pi})\left(1 + O((\log T)^{-1/2+\epsilon})\right)$. By Lemma 6 *(iii)*

$$\left|\frac{\mathcal{S}_1}{\mathcal{S}_2}\right| \ge (1/2)\log(\tfrac{T}{2\pi})\frac{\mathcal{Q}_1}{\mathcal{Q}_2}(1+o(1)) \ge \exp\left(\sqrt{\tfrac{\log M}{\log\log M}}(1+o(1))\right)$$

and thus we establish Theorem 2. □

It suffices to prove Lemma 7.

Proof of Lemma 7. We proceed to bound the various Σ_i. We begin with

$$\Sigma_i = \sum_{un \le M} \frac{f(u)f(nu)(\log n)^i}{\sqrt{n}}$$

for $i = 1, 2$. We evaluate this by writing $(\log n)^i = \sum_{k|n} \Lambda_i(k)$. Inserting this expression we obtain

$$\Sigma_i = \sum_{k \le M} \frac{\Lambda_i(k)f(k)}{\sqrt{k}} \sum_{\substack{nu \le M/k \\ (nu,k)=1}} \frac{f(u)f(nu)}{\sqrt{n}} \le \sum_{k \le M} \frac{\Lambda_i(k)f(k)}{\sqrt{k}} \sum_{\substack{n \le M/k \\ (n,k)=1}} \frac{f(n)}{\sqrt{n}} \sum_{\substack{u \ge 1 \\ (u,kn)=1}} f(u)^2$$

$$\le \sum_{k \le M} \frac{\Lambda_i(k)f(k)}{\sqrt{k}} \sum_{\substack{n \le M/k \\ (n,k)=1}} \frac{f(n)}{\sqrt{n}} \prod_{(p,kn)=1} (1 + f(p)^2) = \mathcal{Q}_2 \sum_{k \le M} \frac{\Lambda_i(k)f(k)}{\sqrt{k}} \sum_{\substack{n \le M/k \\ (n,k)=1}} \frac{f(n)}{\sqrt{n}g(kn)}$$

$$\le \mathcal{Q}_2 \sum_{k \le M} \frac{\Lambda_i(k)f(k)}{\sqrt{k}g(k)} \sum_{n=1}^{\infty} \frac{f(n)}{\sqrt{n}g(n)} = \mathcal{Q}_2 \prod_p \left(1 + \frac{f(p)}{\sqrt{p}g(p)}\right) \sum_{k \le M} \frac{\Lambda_i(k)f(k)}{\sqrt{k}g(k)} \ .$$

The expression in front of the last sum is clearly $\mathcal{Q}_1$. Thus by Lemma 6 (*iv*)

$$\Sigma_1 \ll \mathcal{Q}_1 (\log T)^{1/2+\epsilon} \text{ and } \Sigma_2 \ll \mathcal{Q}_2 (\log T)^{1+\epsilon} \ .$$

Next note that $(\Lambda * \log)(r) \le (\log r)^2$ and hence $\Sigma_3 \le \Sigma_2 \ll (\log T)^{1+\epsilon}$. Next we estimate Σ_4:

$$\begin{aligned}
\Sigma_4 &= \sum_{\substack{a,b \le M \\ (a,b)=1}} \frac{r_1(a,b)}{\sqrt{ab}} \sum_{g \le \min(\frac{M}{a}, \frac{M}{b})} f(ag)f(bg) \\
&\le \mathcal{Q}_1 \sum_{\substack{a,b \le M \\ (a,b)=1}} \frac{f(a)f(b)|r_1(a,b)|}{\sqrt{ab}g(a)g(b)} \\
&\ll \mathcal{Q}_1 \left(\sum_{v \le M} \frac{x_v}{\sqrt{v}g(v)} \right) \left(\sum_{a \le M} \frac{\Lambda_2(a)f(a)}{\sqrt{a}g(a)} + \log T \sum_{a \le M} \frac{\Lambda(a)f(a)}{\sqrt{a}g(a)} \right) \\
&\le \prod_p \left(1 + \frac{f(p)}{\sqrt{p}} + f(p)^2 \right) (\log T)^{3/2+\epsilon}
\end{aligned}$$

by Lemma 6 (*iv*). Finally we have

$$\Sigma_6 = \sum_{ur \le M} \frac{\Lambda(r)f(u)f(ur)}{\sqrt{r}} = \sum_{r \le M} \frac{\Lambda(r)f(r)}{\sqrt{r}} \sum_{\substack{u \le M/r \\ (u,r)=1}} f(u)^2 \le \prod_p (1+x_p^2) \sum_{r \le M} \frac{\Lambda(r)f(r)}{\sqrt{r}g(r)} \ .$$

Once again by Lemma 6 (*iv*) we obtain $\Sigma_6 \ll \mathcal{Q}_2 \ll (\log T)^{1/2+\epsilon}$. $\square$

6. Small values of $\zeta'(\rho)$: Proof of Theorem 3.

Proof of Theorem 3. We begin by noting that Theorem 3 is automatically true if there are infinitely many multiple zeros. Now assume that there are only finitely many multiple zeros of $\zeta(s)$. Suppose there exists a positive constant C' such that for all $\gamma > C'$ all zeros of the zeta function are simple. We will now show that for each T sufficiently large that there exists a $\gamma \in [T, 2T]$ such that

$$|\zeta'(\rho)|^{-1} \geq \exp\left(c_5(1+o(1))\sqrt{\tfrac{\log T}{\log\log T}}\right) \quad (13)$$

and Theorem 3 follows. We now establish (13). Consider the sums

$$S_3 = \sum_{T_1<\gamma<T_2} \zeta'(\rho)^{-1}|A(\rho)|^2 \text{ and } S_2 = \sum_{T_1<\gamma<T_2} |A(\rho)|^2$$

where $A(s) = \sum_{k\leq M} x_k k^{-s}$ and x_k is an arbitary real sequence. Here we choose T_1, T_2 such that

$$\zeta(\sigma + iT_j)^{-1} \ll T_j^{\epsilon}$$

where $T_1 = T + O(1)$ and $T_2 = 2T + O(1)$. This is possible by Theorem 14.16 of [17]. We shall establish:

Proposition 8. *Assume the Riemann hypothesis and that all but finitely many of the zeros of the Riemann zeta function are simple. If* $||\frac{x_n}{n}||_1 \ll T^{\epsilon}$

$$S_3 = \frac{T_2-T_1}{2\pi}\sum_{hn\leq M}\frac{\mu(n)x_h x_{nh}}{nh} + O\left(T^{\epsilon}(M||x_n||_\infty + ||x_n||_1 + T^{\frac{1}{2}}||x_n^2||_1^{\frac{1}{2}})\right)$$

for T sufficiently large.

Moreover by Proposition 4 we have

$$S_2 = (N(T_2)-N(T_1))\sum_{m\leq M}\frac{x_m^2}{m} - \frac{T_2-T_1}{\pi}\sum_{m\leq M}\frac{(\Lambda * x)(m)x_m}{m} + o(T)$$

respectively. We now choose $x_m = \sqrt{m}\mu(m)f(m)$ and suppose that $M < T^{2/3-10\epsilon}$. Note that $||x_n||_\infty \ll M^{\frac{1}{2}+\epsilon}, ||x_n||_1 \ll M^{1+\epsilon}$ and thus

$$S_3 = \frac{T_2-T_1}{2\pi}\left(\sum_{hn\leq M}\frac{f(h)f(nh)}{\sqrt{n}} + o(1)\right)$$

and

$$S_2 = (N(T_2)-N(T_1))\sum_{m\leq M} f(m)^2 - \frac{T_2-T_1}{\pi}\sum_{m\leq M}\frac{(\Lambda * f)(m)f(m)}{m} + o(T)\ .$$

The second sum in S_2 is bounded by

$$\sum_{mp\leq M}\frac{(\log p)f(m)f(mp)}{\sqrt{p}} \ll \sum_{p\leq b}\frac{\log p f(p)}{\sqrt{p}}\sum_{\substack{m\leq M/p\\(m,p)=1}} f(m)^2 \ll \left(\sum_{m\leq M} f(m)^2\right)(\log T)^{1/2+\epsilon}\ .$$

With these observations in hand we obtain

$$\max_{T\leq\gamma\leq 2T}|\zeta'(\rho)|^{-1} \geq \log(\tfrac{T}{2\pi})^{-1}\left(\frac{\sum_{hn\leq M}\frac{f(h)f(nh)}{\sqrt{n}}}{\sum_{m\leq M} f(m)^2}\right)(1+o(1))$$

and by Soundararajan's calculation we obtain

$$\max_{T\leq\gamma\leq 2T}|\zeta'(\rho)|^{-1} \geq \exp\left((1+o(1))\sqrt{\tfrac{\log M}{\log\log M}}\right)$$

for $M < T^{\frac{2}{3}-10\epsilon}$ which yields (13).

It now suffices to establish Proposition 8.

Proof of Proposition 8. We consider the integral

$$I := \frac{1}{2\pi i}\int_{c+iT_1}^{c+iT_2} \zeta(s)^{-1}A(s)A(1-s)\,ds\ .$$

with $c = 1 + O((\log T)^{-1})$. Moving the contour left to the $1-c$ line yields $I = S_3 + H + I' + O(1)$ where

$$I' := \frac{1}{2\pi i}\int_{1-c+iT_1}^{1-c+iT_2} \zeta(s)^{-1}A(s)A(1-s)\,ds$$

and H are the horizontal contributions. We know from Proposition 4 that

$$I = \frac{T_2-T_1}{2\pi}\sum_{nu\le M}\frac{\mu(n)x_u x_{nu}}{nu} + O(M^{\epsilon}(||x_n||_{\infty}M + ||x_n||_1))\ .$$

Next we consider the contribution from the horizontal terms. We may verify that $|A(s)A(1-s)| \le M||\frac{x_n}{n}||_1^2 + ||x_n||_1||\frac{x_n}{n}||_1$ for $1-c \le \mathrm{Re}(s) \le c$. Furthermore, since we have chosen the T_j such that $\zeta(\sigma+iT_j)^{-1} \ll T_j^{\epsilon}$, $H \ll T^{\epsilon}(M + ||x_n||_1)$. We now consider the contribution of the left hand side. We have that $\zeta(s) = \chi(s)\zeta(1-s)$. Since $\chi(s) \asymp T^{1/2}$ and $\zeta(1-s) \asymp \log T$ for $\mathrm{Re}(s) = 1-c$ we have

$$\begin{aligned} I' &\ll (T^{1/2}\log T)^{-1}||\tfrac{x_n}{n}||_1\int_{T_1}^{T_2}|A(1-c+it)|\,dt \\ &\ll (\log T)^{-1}||\tfrac{x_n}{n}||_1\left(\int_{T_1}^{T_2}|A(1-c+it)|^2\,dt\right)^{1/2}\ . \end{aligned}$$

The mean value theorem for Dirichlet polynomials asserts

$$\int_{T_1}^{T_2}\left|\sum_{n\le N}\frac{a_n}{n^{it}}\right|^2 dt = (T_2-T_1)\sum_{n\le N}|a_n|^2 + O\Big(\textstyle\sum_{n\le N} n|a_n|^2\Big)\ .$$

Since $1-c = O((\log T)^{-1})$

$$\int_{T_1}^{T_2}|A(1-c+it)|^2\,dt \ll T\sum_{n\le M}x_n^2 + \sum_{n\le M}\frac{x_n^2}{n} \ll T||x_n^2||_1\ .$$

Thus we deduce that $I' \ll T^{\frac{1}{2}}(\log T)^{-1}||\frac{x_n}{n}||_1||x_n^2||_1^{\frac{1}{2}}$. Collecting estimates yields Proposition 8. □

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