THE MÖBIUS FUNCTION IN SHORT INTERVALS

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Abstract. In this article we consider $M(x)$, the summatory function of the Möbius function, in short intervals. More precisely, we give an argument which suggests that $M(x+h)-M(x)$ for $0 \le x \le N$ is approximately normal with mean ~ 0 and variance $\sim \frac{6h}{\pi^2}$ where h and N satisfy appropriate conditions. This argument is conditional on the assumption of a version of the Hardy-Littlewood prime k-tuples conjecture adapted to the case of the Möbius function.

1. Introduction

Let $\mu(n)$ denote the Möbius function, a multiplicative function supported on squarefree integers. We have $\mu(1)=1$ and for $n=p_1\ldots p_k>1$ squarefree we have $\mu(n)=1$ if k is even and $\mu(n)=-1$ if k is odd. A well-studied function is the summatory function

$$M(x)=\sum_{n\le x}\mu(n) \ .$$

It plays an important role in analytic number theory since many questions pertaining to primes can be rephrased in terms of $M(x)$. For example, the prime number theorem is equivalent to showing that $M(x)=o(x)$ and the Riemann hypothesis is equivalent to showing that $M(x) \ll x^{\frac{1}{2}+\epsilon}$.

It seems reasonable to expect that the distribution of values of $M(x)$ behaves like the distribution of values of a function, which is zero on non-squarefree integers, and whose value ie either -1 or 1 on squarefree integers, the choice of -1 or 1 being made randomly for each integer. More precisely, a model for $M(x)$ is the function $M_{rand}(x)=\sum_{n\le x}' X_n$ where the sum Σ' is restricted to squarefree integers and the X_n are a sequence of independent identically distributed random variables such that $X_n=1$ with probability $1/2$ and $X_n=-1$ with probability $1/2$. The variance of $M_{rand}(x)$ is $\sum_{n\le x}' 1=\sum_{n\le x}\mu^2(n)=\frac{6x}{\pi^2}(1+o(1))$ and therefore by the central limit theorem the distribution function of $M_{rand}(x)/\sqrt{(6x/\pi^2)}$ is the normal distribution $\Phi(c):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-\frac{\xi^2}{2}}\,d\xi$.

In [12] we studied $\mu(n)$ in the long interval $[1,x]$. Assuming RH, we have the explicit formula

$$M(x)x^{-1/2}=2\mathrm{Re}\Big(\sum_{\gamma>0}\frac{x^{i\gamma}}{\rho\zeta'(\rho)}\Big)$$

where $\rho=1/2+i\gamma$ ranges over non-trivial zeros of the zeta function (see [13] pages 372-374). Assuming RH and the bound $\sum_{0<\gamma<T}|\zeta'(\rho)|^{-2} \ll T$, the author proved the existence of a limiting distribution for $M(e^y)e^{-y/2}$. Surprisingly, this

distribution is not the normal distribution as was suggested by the above analogy between $M(x)$ and $M_{rand}(x)$. If we possess some very fine information concerning the behaviour of the imaginary ordinates of the zeros of $\zeta(s)$ then this distribution can be described rather explicitly. There is some evidence to suggest that the imaginary ordinates of the non-trivial zeros of $\zeta(s)$ do not satisfy any $\mathbb{Q}$-linear relations. Under this additional assumption, the limiting distribution of $M(e^y)e^{-y/2}$ agrees with the distribution of the random sum

$$\mathrm{Re}\Big(\sum_{\gamma>0}\frac{X(\gamma)}{\rho\zeta'(\rho)}\Big) \tag{1}$$

where the $X(\gamma)$ are independent random variables uniformly distributed on the unit circle. For further analysis of distributions of this type see [9].

In this note, we shall investigate the behaviour of $M(x)$ in shorter intervals. More precisely, we are concerned with the distribution of

$$M(n+h)-M(n)=\sum_{1\le m\le h}\mu(n+m)$$

where $1\le n\le N$ and $h\le N$. As above, we may model this by

$$M_{rand}(n+h)-M_{rand}(n)=\textstyle\sum'_{1\le m\le h}X_{n+m}$$

where we recall that Σ' means that we restrict to squarefree integers in the range of summation. However, in this setting $M(n+h)-M(n)$ is correctly modelled by its random version. In fact, this type of reasoning was previously considered by Good and Churchhouse [7] who made the following conjecture:

Conjecture A. The sums of $\mu(n)$ in blocks of length h, where h is large, have asymptotically a normal distribution with mean zero and variance $\frac{6h}{\pi^2}$.

The goal of this note is to provide some theoretical evidence supporting the above conjecture. In order to determine the distribution of $M(n+h)-M(n)$ we shall apply the moment method. We will calculate the moments

$$\nu_k(N;h)=\sum_{n\le N}(M(n+h)-M(n))^k$$

and thus deduce a distribution result. This is a well-known argument and has recently been employed in [8] and [11]. In our analysis of $\nu_k(N;h)$ we will assume the following conjecture concerning the Möbius function.

Möbius s-tuple conjecture. Let $s\in\mathbb{N}$ and $\mathcal{D}=\{d_1,\ldots,d_s\}$ denote s distinct integers with $\alpha_1,\ldots,\alpha_s\in\mathbb{N}$. If at least one α_i is odd then there exists $\frac{1}{2}<\beta_0<1$ independent of s such that

$$\sum_{n\le N}\mu(n+d_1)^{\alpha_1}\ldots\mu(n+d_s)^{\alpha_s}\ll N^{\beta_0} \tag{2}$$

uniformly for all $|d_i|\le N$.

Note that when $|\mathcal{D}|=1$ the Riemann hypothesis implies that $\beta_0=\frac{1}{2}+\epsilon$ is an admissible value. For larger s this is related to a conjecture for s-tuples of primes. Let $\mathcal{D}=\{d_1,\ldots,d_s\}$ denotes a set of s distinct integers. Hardy and Littlewood conjectured that

$$\sum_{n\le x}\prod_{i=1}^{s}\Lambda(n+d_i)=(\mathfrak{S}(\mathcal{D})+o(1))x \tag{3}$$

where

$$\mathfrak{S}(\mathcal{D})=\prod_p\left(1-\frac{1}{p}\right)^{-k}\left(1-\frac{\tilde{\nu}(p;\mathcal{D})}{p}\right)$$

is the singular series attached to $\mathcal{D}$ and $\tilde{\nu}(p;\mathcal{D})$ denotes the number of distinct residue classes modulo p among all members of $\mathcal{D}$. In [11] Montgomery and Soundararajan studied the distribution of $\psi(x)=\sum_{n\le x}\Lambda(n)$ in short intervals by assuming a version of conjecture (3) with the error term $o(1)$ replaced by $O(x^{-1/2+\epsilon})$. They determined that distribution of $\psi(n+h)-\psi(n)$ for $1\le n\le N$ is approximately normal with mean $\sim h$ and variance $\sim h\log(N/h)$ for an appropriate range of h and N.

We now state our results for $\mu(n)$ in short intervals. For k a natural number we introduce the notation

$$C_k=\begin{cases}\frac{\Gamma(k+1)}{2^{k/2}\Gamma(k/2+1)} & \text{if } k \text{ is even}\\ 0 & \text{if } k \text{ odd}\end{cases}.$$

By following the argument of [11] we obtain

Theorem 1. *Let k be a natural number. Assume the Möbius s-tuple conjecture (2) holds uniformly for $1\le s\le k$ (i.e. there exists a β_0 independent of s such that (2) holds for all $1\le s\le k$). If k is even then we have*

$$\nu_k(N;h)=C_kN\left(\tfrac{6h}{\pi^2}\right)^{k/2}(1+O((\log h)^kh^{-1/2}+k^3h^{-1}))+O_k(N^{\max(\beta_0,2/3)}h^k) \tag{4}$$

uniformly for $h=o(N^{\frac{2}{k}(1-\max(\beta_0,\frac{2}{3}))})$ and $k\le h^{1/3}$. If k is odd

$$\nu_k(N;h)=O_k(N^{\beta_0}h^k)\ . \tag{5}$$

uniformly for $h=o(N^{\frac{2}{k}(1-\beta_0)})$.

Observe that the main term is the k-th moment of a normal random variable with expectation 0 and variance $\frac{6h}{\pi^2}$. We remark that the first O term in (4) is *independent* of k whereas the second one depends on k. In order to remove this dependence on k we would have to formulate an appropriate version of the Möbius s-correlation conjecture with an explicit dependence on s. By a familiar argument we deduce

Theorem 2. *Let $h=h(N)\to\infty$ such that $\frac{\log h}{\log N}\to 0$ as $N\to\infty$. Assume the Möbius s-tuple conjecture holds for arbitrarily large s. Then the distribution of $M(n+h)-M(n)$ for $n\le N$ is approximately normal with mean ~ 0 and variance $\sim\frac{6h}{\pi^2}$. More precisely,*

$$\frac{1}{N}\#\{1\le n\le N\mid M(n+h)-M(n)\le c\sqrt{\tfrac{6h}{\pi^2}}\ \}\to\Phi(c)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c}e^{-x^2/2}\,dx$$

uniformly for $|c|\le C$ where C is a fixed positive real number.

This proof may be obtained by an application of the Berry-Esseen theorem (for a reference to this type of argument see [3]). Note that the above theorem furnishes a distribution result for a rather restricted range of h. We actually believe that it should continue to be true for $h \leq N^{1-\epsilon}$.

Conjecture B. For each positive integer k,

$$\nu_k(N;h) = (C_k + o(1))N(\tfrac{6h}{\pi^2})^{k/2}$$

uniformly for $h(N) \leq h \leq N^{1-\epsilon}$ where $h(N) \to \infty$.

This leads us to formulate a more precise version of Conjecture A.

Conjecture A′. Suppose that $h(N) \leq h \leq N^{1-\epsilon}$ where $h(N) \to \infty$. The distribution of $M(n+h) - M(n)$ for $0 \leq n \leq N$ is approximately normal with mean ~ 0 and variance $\sim \frac{6h}{\pi^2}$.

2. Calculation of the moments: Proof of Theorem 1

Proof of Theorem 1. We begin by assuming that k is even. Writing $M(n+h) - M(n) = \sum_{1 \leq m \leq h} \mu(n+m)$ it follows that

$$\nu_k(N;h) = \sum_{\substack{m_1,\ldots,m_k \\ 1 \leq m_i \leq h}} \sum_{n \leq N} \mu(n+m_1)\mu(n+m_2)\ldots\mu(n+m_k) \ .$$

Suppose that given $\{m_1, \ldots, m_k\}$ integers in the box $[1,h]^k$ that $\{d_1, \ldots, d_s\}$ denote the distinct integers in this set with multiplicites $\alpha_1, \ldots, \alpha_s$ such that $\sum_{i=1}^{s} \alpha_i = k$. Thus we have

$$\nu_k(N;h) = \sum_{s=1}^{k} \sum_{\substack{\alpha_1,\ldots,\alpha_s \\ \sum_i \alpha_i = k}} \binom{k}{\alpha_1, \ldots, \alpha_s} \frac{1}{s!} \sum_{\substack{d_1,\ldots,d_s \\ 1 \leq d_i \leq h}} \sum_{n \leq N} \mu(n+d_1)^{\alpha_1} \ldots \mu(n+d_s)^{\alpha_s} \ . \quad (6)$$

Next we note that $\mu(n)^{\alpha} = \mu(n)$ if α is odd and $\mu(n)^{\alpha} = \mu(n)^2$ if α is even. By the Möbius s-tuple conjecture, those s-tuples $\{\alpha_1, \ldots, \alpha_s\}$ with at least one odd member will contribute an error term $O(N^{\beta_0})$. Therefore, the principal term will arise from the s-tuples with all α_i even. In this case, we invoke the following strong theorem of Tsang [14]:

Proposition 3. *Let $\mathcal{D} = \{d_1, \ldots, d_s\}$ be distinct integers such that $|d_i| \leq N$ and $s \leq \frac{\log N}{25 \log\log N}$. Then*

$$\sum_{n \leq N} \mu(n+d_1)^2 \mu(n+d_2)^2 \ldots \mu(n+d_s)^2 = NA(\mathcal{D}) + o(N^{2/3})$$

where

$$A(\mathcal{D}) = A(d_1, \ldots, d_s) = \prod_p \left(1 - \frac{\nu(p;\mathcal{D})}{p^2}\right)$$

and

$$\nu(p;\mathcal{D}) = \#\{ \ a \bmod p^2 \ | \ \exists \ d \in \mathcal{D} \text{ such that } a \equiv d \pmod{p^2}\} \ .$$

Tsang obtains this theorem by an application of the combinatorial sieve. It is important to note that the little o term is completely independent of s. It would be interesting to reduce the exponent $\frac{2}{3}$.

Combining these last observations we obtain

$$\begin{aligned}\nu_k(N;h) = N\sum_{s=1}^{k}\sum_{\substack{\alpha_1,\ldots,\alpha_s\geq 1\\ \sum_i \alpha_i=k\\ \alpha_i \text{ even}}}\binom{k}{\alpha_1,\ldots,\alpha_s}\frac{1}{s!}\sum_{\substack{d_1,\ldots,d_s\\ 1\leq d_i\leq h}}A(d_1,\ldots,d_s)\\ + o(N^{2/3}\sum_{s=1}^{k}\sum_{\substack{\alpha_1,\ldots,\alpha_s\geq 1\\ \sum_i \alpha_i=k\\ \alpha_i \text{ even}}}\binom{k}{\alpha_1,\ldots,\alpha_s}\frac{1}{s!}\sum_{\substack{d_1,\ldots,d_s\\ 1\leq d_i\leq h}}1)\\ + O(N^{\beta_0}\sum_{s=1}^{k}\sum_{\substack{\alpha_1,\ldots,\alpha_s\geq 1\\ \sum_i \alpha_i=k\\ \text{one } \alpha_i \text{ odd}}}\binom{k}{\alpha_1,\ldots,\alpha_s}\frac{1}{s!}\sum_{\substack{d_1,\ldots,d_s\\ 1\leq d_i\leq h}}1)\ .\end{aligned}$$

We now define $\beta_1 = \max(\beta_0, 2/3)$. Note that the two error terms combined are bounded by

$$N^{\beta_1}\sum_{s=1}^{k}\sum_{\substack{\alpha_1,\ldots,\alpha_s\geq 1\\ \sum_i \alpha_i=k}}\binom{k}{\alpha_1,\ldots,\alpha_s}\frac{1}{s!}\sum_{\substack{d_1,\ldots,d_s\\ 1\leq d_i\leq h}}1 = N^{\beta_1}h^k$$

and thus

$$\nu_k(N;h) = N\sum_{s=1}^{k}\sum_{\substack{\alpha_1,\ldots,\alpha_s\geq 1\\ \sum_i \alpha_i=k\\ \alpha_i \text{ even}}}\binom{k}{\alpha_1,\ldots,\alpha_s}\frac{1}{s!}\sum_{\substack{d_1,\ldots,d_s\\ 1\leq d_i\leq h}}A(d_1,\ldots,d_s) + O_k(N^{\beta_1}h^k)\ .$$

To complete our calculation we shall establish:

Proposition 4.

$$\sum_{\substack{1\leq d_1,\ldots,d_s\leq h\\ d_i \text{ distinct}}}A(d_1,\ldots,d_s) = \left(\frac{6}{\pi^2}\right)^s h^s(1+O((\log h)^s h^{-1/2}))\ .$$

We will postpone the proof of this proposition until the next section. With Proposition 4 in hand we have

$$\begin{aligned}\nu_k(N;h) = N\sum_{s=1}^{k}\sum_{\substack{\alpha_1,\ldots,\alpha_s\geq 1\\ \sum_i \alpha_i=k\\ \alpha_i \text{ even}}}\binom{k}{\alpha_1,\ldots,\alpha_s}\frac{1}{s!}\left(\frac{6h}{\pi^2}\right)^s(1+O((\log h)^s h^{-1/2}))\\ + O_k(N^{\max(\beta_1)}h^k)\ .\end{aligned}$$

We begin my remarking that the conditions in the sum force $s \leq k/2$ since all α_i are even. The term with $s = \frac{k}{2}$ contributes

$$C_k\left(\frac{6h}{\pi^2}\right)^{k/2}\left(1+O\left(\frac{(\log h)^k}{h^{1/2}}\right)\right)\ .$$

The terms with $s < k/2$ are bounded by

$$\sum_{s<k/2} \frac{k!}{s!} \left(\frac{6h}{\pi^2}\right)^s \sum_{\substack{\alpha_1,\dots,\alpha_s \ge 2 \\ \sum_i \alpha_i = k \\ \alpha_i \text{ even}}} \frac{1}{\alpha_1! \cdots \alpha_k!} \ .$$

The number of ways of writing $k = \alpha_1 + \cdots + \alpha_s$ with each $\alpha_i \ge 2$ equals the number of ways of writing $k - s = \alpha_1' + \cdots + \alpha_s'$ where each $\alpha_i' \ge 1$ and thus equals $\binom{k-s}{s}$. The remaining terms are therefore bounded by

$$\sum_{s<k/2} \frac{k!}{s! 2^s} \binom{k-s}{s} h^s \ll C_k h^{k/2-1} k^3$$

if we assume that $k \le h^{1/3}$. Thus we have shown that

$$\nu_k(N;h) = C_k N \left(\frac{6h}{\pi^2}\right)^{k/2} (1 + O((\log h)^k h^{-1/2} + k^3 h^{-1})) + O_k(N^{\beta_1} h^k)$$

assuming k is even. When k is odd the same argument works. However, in the expansion (6) we always have at least one α_i odd since k is odd. Thus no main term emerges in this case and the inner sum is bounded by $O(N^{\beta_0})$ for all choices of indices and we thus obtain $\nu_k(N,h) \ll_k h^k N^{\beta_0}$. □

3. Proof of Proposition 4

Our argument for proving Proposition 4 follows Gallagher's method [2] for evaluating

$$\sum_{1 \le d_1,\dots,d_s \le h} \mathfrak{S}(d_1,\dots,d_s) \ .$$

This argument provides a savings of $O(h^{-1/2+\epsilon})$ from the main term. In [11] a more sophisticated argument is applied which gives a savings of $O(h^{-1+\epsilon})$. However, this is not required for our purposes.

Proof of Proposition 4. We write

$$A(\mathcal{D}) = \prod_p \left(1 - \frac{\nu(p;\mathcal{D})}{p^2}\right) = \sum_{n\ge 1} \frac{\mu(n)\nu(n;\mathcal{D})}{n^2}$$

where we define for squarefree n, $\nu(n;\mathcal{D}) = \prod_{p|n} \nu(p;\mathcal{D})$. In this argument we shall apply repeatedly the bounds

$$\sum_{n\le x} \frac{s^{\omega(n)}}{n} \ll (\log x)^s \text{ and } \sum_{n \le x} s^{\omega(n)} \le x \sum_{n\le x} \frac{s^{\omega(n)}}{n} \ll x(\log x)^s \ .$$

Since $|\nu(p;\mathcal{D})| \le s$, it follows that $|\nu(n;\mathcal{D})| \le s^{\omega(n)}$ and thus

$$\sum_{n>x} \frac{\mu(n)\nu(n;\mathcal{D})}{n^2} \ll \sum_{n>x} \frac{s^{\omega(n)}}{n^2} \ll \frac{(\log x)^s}{x} \ .$$

Set $a(n;\mathcal{D}) = \mu(n)\nu(n;\mathcal{D})/n^2$ and it follows that

$$\sum_{\substack{1\le d_1,\dots,d_s\le h \\ d_i \text{ distinct}}} A(d_1,\dots,d_s) = \sum_{n\le x} \sum_{\substack{1\le d_1,\dots,d_s\le h \\ d_i \text{ distinct}}} a(n;\mathcal{D}) + O\left(\frac{h^s(\log x)^s}{x}\right) \ .$$

Suppose that n is a fixed squarefree integer $\le x$ and $n = p_1 \dots p_t$ (i.e. $\omega(n) = t$). For each $1 \le i \le t$ let ν_i be a variable satisfying $1 \le \nu_i \le p_i^2$ and $\vec{\nu} = (\nu_1, \dots, \nu_t) \in \mathbb{N}^t$. It follows that

$$\sum_{\substack{1 \le d_1, \dots, d_s \le h \\ d_i \text{ distinct}}} a(n; \mathcal{D}) = \sum_{\substack{\vec{\nu} = (\nu_1, \dots, \nu_t) \\ 1 \le \nu_i \le p_i^2}} \prod_{i=1}^{t} a(p_i; \nu_i) \Big(\sum_{\substack{\mathcal{D} = (d_1, \dots, d_s) \\ 1 \le d_i \le h \\ \forall p_i | n \ \nu(p_i, \mathcal{D}) = \nu_i}} 1 + O(h^{s-1}) \Big) \tag{7}$$

where $a(p; \nu) = \frac{\mu(p)\nu}{p^2}$. By a lattice point argument employing the Chinese remainder theorem the inner sum is

$$\left(\left(h/n^2 \right)^s + O\left(\left(h/n^2 \right)^{s-1} \right) \right) \prod_{i=1}^{t} \binom{p_i^2}{\nu_i} \sigma(s, \nu_i)$$

where we assume that $n \le \sqrt{h}$ and $\sigma(s, \nu)$ denotes the number of maps from $\{1, \dots, s\}$ onto $\{1, \dots, \nu\}$. Inserting this last expression in (7) we obtain

$$\sum_{\substack{1 \le d_1, \dots, d_s \le h \\ d_i \text{ distinct}}} a(n; \mathcal{D}) = (h/n^2)^s \alpha(n) + O((h/n^2)^{s-1} \beta(n)) + O(h^{s-1} \gamma(n))$$

where

$$\alpha(n) = \sum_{\vec{\nu}} \prod_{i=1}^{t} a(p_i; \nu_i) \binom{p_i^2}{\nu_i} \sigma(s, \nu_i) = \prod_{p|n} \sum_{v=1}^{p^2} a(p; \nu) \binom{p^2}{\nu} \sigma(s, \nu) \ ,$$

$$\beta(n) = \sum_{\vec{\nu}} \prod_{i=1}^{t} |a(p_i; \nu_i)| \binom{p_i^2}{\nu_i} \sigma(s, \nu_i) = \prod_{p|n} \sum_{v=1}^{p^2} |a(p; \nu)| \binom{p^2}{\nu} \sigma(s, \nu) \ ,$$

$$\gamma(n) = \sum_{\vec{\nu}} \prod_{i=1}^{t} |a(p_i; \nu_i)| = \prod_{p|n} \sum_{v=1}^{p^2} |a(p; \nu)| \ .$$

We now estimate $\alpha(n), \beta(n)$, and $\gamma(n)$ on average over n. Since $|a(p; \nu)| \le s/p^2$ and $|\gamma(n)| \le s^{\omega(n)} n^{-1}$ it follows that

$$\sum_{n \le x} \gamma(n) \ll (\log x)^s \ .$$

Similarly, by the identity $\sum_{v=1}^{p^2} \binom{p^2}{v} \sigma(s, v) = p^{2s}$, we obtain

$$|\beta(n)| \le \prod_{p|n} \frac{s}{p^2} \sum_{v=1}^{p^2} \binom{p^2}{\nu} \sigma(r, \nu) \le \prod_{p|n} \frac{s}{p^2} p^{2s} = s^{\omega(n)} n^{2s-2}$$

and thus

$$\sum_{n \le x} \frac{\beta(n)}{n^{2(s-1)}} \ll \sum_{n \le x} s^{\omega(n)} \ll x (\log x)^s \ .$$

Similarly, it may be checked that $|\alpha(n)| \le s^{\omega(n)} n^{2s-2}$ and thus

$$\sum_{n > x} \frac{\alpha(n)}{n^{2s}} \ll \sum_{n > x} \frac{s^{\omega(n)}}{n^2} \ll \frac{(\log x)^s}{x} \ .$$

Collecting estimates yields

$$\sum_{\substack{1\le d_1,\ldots,d_s\le h\\ d_i \text{ distinct}}} A(d_1,\ldots,d_s) = h^s\sum_{n=1}^{\infty}\frac{\alpha(n)}{n^{2s}} + O\left(h^s\sum_{n>x}\frac{\alpha(n)}{n^{2s}} + h^{s-1}x(\log x)^s\right) .$$

As $\alpha(n)$ is a multiplicative function

$$\sum_{n=1}^{\infty}\frac{\alpha(n)}{n^{2s}} = \prod_p\left(1+\frac{\alpha(p)}{p^{2s}}\right) .$$

Since $a(p,\nu) = -\nu/p^2$ we have

$$\frac{\alpha(p)}{p^{2s}} = -\frac{1}{p^{2s+2}}\sum_{\nu=1}^{p^2}\nu\binom{p^2}{\nu}\sigma(s,\nu) .$$

However, by the identity

$$\sum_{v=1}^{p^2}\nu\binom{p^2}{\nu}\sigma(s,v) = (p^2)^{s+1} - (p^2-1)^s p^2$$

and it follows that

$$\frac{\alpha(p)}{p^{2s}} = -\frac{1}{p^{2s+2}}(p^{2s+2}-(p^2-1)^s p^2) = -1+\left(1-\frac{1}{p^2}\right)^s .$$

Therefore $\sum_{n=1}^{\infty}\alpha(n)n^{-s} = \zeta(2)^{-2} = (\frac{6}{\pi^2})^s$ and we deduce that

$$\sum_{\substack{1\le d_1,\ldots,d_s\le h\\ d_i \text{ distinct}}} A(d_1,\ldots,d_s) = \left(\frac{6h}{\pi^2}\right)^s + O\left(\frac{h^s(\log x)^s}{x} + h^{s-1}x(\log x)^s\right)$$

which is valid for $x\le\sqrt{h}$. The choice $x=\sqrt{h}$ completes the proof. □

4. Equivalent formulations

In this section we present several equivalent formulations of the asymptotic formula for $\nu_k(N;h)$. It follows from Theorem 1 that for k even the Möbius randomness conjecture implies that

$$\int_1^X (M(x+h)-M(x))^k\,dx \sim \mu_k X(6h/\pi^2)^{k/2}$$

for $h(X)\le h\le X^{c_0/k}$ for an appropriate $c_0>0$ and $h(X)\to\infty$ as $X\to\infty$. However, it is plausible that this actually holds in the larger interval $h(X)\le h\le X^{1-\epsilon}$. We have the following variant of the above asymptotic formula.

Proposition 5. *Let k be a positive even integer. Assume the Riemann hypothesis. The following statements are equivalent:*

$$\int_1^X (M(x+h)-M(x))^k\,dx \sim \mu_k X(6h/\pi^2)^k \tag{8}$$

holds uniformly for $X^{\epsilon}\le h\le X^{1-\epsilon}$.

$$\int_1^X (M(x+\delta x)-M(x))^k\,dx \sim \frac{\mu_k}{k/2+1}X^{k/2+1}(6\delta/\pi^2)^{k/2} \tag{9}$$

holds uniformly for $X^{-1+\epsilon} \le \delta \le X^{-\epsilon}$.

The proof follows an argument given by Chan [1]. He proved similar identities relating integrals of $\psi(x+h)-\psi(x)-h$ to integrals of $\psi(x+\delta x)-\psi(x)-\delta x$.

We now mention some work related to the conjectured asymptotics (8) and (9). Peng Gao has informed me that he can prove under the assumption of the Riemann hypothesis that for $X \ge 2$ and $h \ge (\log X)^A$ with A explicit and fixed that

$$\int_1^X (M(x+h)-M(x))^2 \, dx = o(Xh^2) \ .$$

Observe that this is slightly stronger than the trivial bound $O(Xh^2)$. In addition, Gonek [5], [6] has some unpublished work concerning the case $k=2$ of (9). He undertook a study of the function

$$G(X,T) = \sum_{0<\gamma,\gamma'<T} X^{i(\gamma-\gamma')} \frac{\omega(\gamma-\gamma')}{\zeta'(\rho)\zeta'(\rho')}$$

for a certain smooth weight ω. This is analogous to Montgomery's pair correlation function

$$F(X,T) = \sum_{0<\gamma,\gamma'<T} X^{i(\gamma-\gamma')}\omega(\gamma-\gamma')$$

where $\omega(u) = 4/(4+u^2)$. In [4] very precise relations between the behaviour of $F(X,T)$ and the second moment of $\psi(x+h)-\psi(x)-h$ are established. In the same fashion Gonek studied the behavioiur of $G(X,T)$. This is more difficult than the study of $F(X,T)$ due to the erratic behaviour of $\zeta'(\rho)^{-1}$. He develops various formulae for $G(X,T)$ assuming the Riemann hypothesis and an upper bound of the form $|\zeta^{'}(\rho)|^{-1} \ll |\rho|^{1/2-\epsilon}$. This led him to conjecture that

$$\int_1^X \left(\frac{M(x+\delta x)-M(x)}{x} \right)^2 dx \sim \frac{6\delta}{\pi^2}(\log X)$$

for $\delta \ge 1/X$. Note that this follows from (9) by partial integration (at least for δ in an appropriate range).

5. Generalizations

Our argument for evaluating $\nu_k(N;h)$ may be generalized considerably. For example, it works for the Liouville function $\lambda(n)$ which is defined to be $\lambda(n) = (-1)^{\Omega(n)}$ where $\Omega(n)$ is the total number of prime factors of n. We define its summatory function to be $L(x) = \sum_{n\le x} \lambda(n)$. In order to compute its moments

$$\nu_k(N,h) = \sum_{n\le N} (L(n+h)-L(n))^k$$

we need to consider

$$\sum_{n\le N} \lambda(n+d_1)^{\alpha_1} \cdots \lambda(n+d_s)^{\alpha_s}$$

where $\{d_1,\dots,d_s\}$ are distinct and $s \le k$. Observe that $\lambda(n)^\alpha = 1$ if α is even and $\lambda(n)^\alpha = \lambda(n)$ if α is odd. Therefore if all α_i are even then the above sum exactly equals N. In addition, we require an analogue of the Möbius s-tuple conjecture for λ.

Liouville s-tuple conjecture. Let $s \in \mathbb{N}$ and $\mathcal{D} = \{d_1, \ldots, d_s\}$ denote s distinct integers with $\alpha_1, \ldots, \alpha_s \in \mathbb{N}$. If at least one α_i is odd then there exists $\frac{1}{2} < \beta_0 < 1$ independent of s such that

$$\sum_{n \le N} \lambda(n+d_1)^{\alpha_1} \ldots \lambda(n+d_s)^{\alpha_s} \ll N^{\beta_0}$$

uniformly for all $|d_i| \le N$.

Assuming the above holds uniformly for $1 \le s \le k$ we see that

$$\nu_k(N,h) = C_k N h^{k/2}(1 + O(k^3 h^{-1})) + O_k(N^{\beta_0} h^k)$$

as long as k is even and $k \le h^{1/3}$. It follows that $L(n+h) - L(n)$ for $1 \le n \le N$ is approximately normal with mean ~ 0 and variance $\sim h$ for $h = h(N)$ which satifies $h \to \infty$ and $\frac{\log h}{\log N} \to 0$.

It appears that the argument applied to $\mu(n)$ and $\lambda(n)$ may be applied to a much wider class of real multiplicative functions f with mean value 0 and satisfying $|f(n)| \le 1$. In order to determine

$$\nu_k(N,h) = \sum_{n \le N} (f(n+h) - f(n))^k$$

we would require a formula for

$$\sum_{n \le N} f(n+d_1)^{\alpha_1} \cdots f(n+d_s)^{\alpha_s}$$

with $\{d_1, \ldots, d_s\}$ distinct and all α_i even. Also, if at least one α_i is odd, we would require a bound of the shape

$$\sum_{n \le N} f(n+d_1)^{\alpha_1} \cdots f(n+d_s)^{\alpha_s} \ll N^{\beta_0} \ .$$

It would be interesting to determine the class of multiplicative functions which satisfies this bound.

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