THE SIXTH MOMENT OF THE RIEMANN ZETA FUNCTION AND TERNARY ADDITIVE DIVISOR SUMS

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ABSTRACT. Hardy and Littlewood initiated the study of the $2k$-th moments of the Riemann zeta function on the critical line. In 1918 Hardy and Littlewood established an asymptotic formula for the second moment and in 1926 Ingham established an asymptotic formula for the fourth moment. Since then no other moments have been asymptotically evaluated. In this article we study the sixth moment of the zeta function on the critical line. We show that a conjectural formula for a certain family of ternary additive divisor sums implies an asymptotic formula with power savings error term for the sixth moment of the Riemann zeta function on the critical line. This provides a rigorous proof for a heuristic argument of Conrey and Gonek [11]. Furthermore, this gives some evidence towards a conjecture of Conrey, Keating, Farmer, Rubinstein, and Snaith [8] on shifted moments of the Riemann zeta function. In addition, this improves on a theorem of Ivic [31], who obtained an upper bound for the the sixth moment of the zeta function, based on the assumption of a conjectural formula for correlation sums of the triple divisor function.

1. INTRODUCTION

The $2k$-th moment of the Riemann zeta function is

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt, \tag{1.1}$$

where ζ denotes the Riemann zeta function and $k > 0$. This article concerns the behaviour of (1.1) in the case $k = 3$. Hardy and Littlewood initiated the study of the moments (1.1). Their interest in these mean values arose from their relation to the Lindelöf hypothesis, which asserts that for any $\varepsilon > 0$ $|\zeta(\frac{1}{2}+it)| \ll_\varepsilon t^\varepsilon$. In fact, they showed the Lindelöf hypothesis is equivalent to the statement, for any $\varepsilon > 0$, $I_k(T) \ll_\varepsilon T^{1+\varepsilon}$ for all $k \in \mathbb{N}$. The motivation for studying the moment $I_k(T)$ is that it seems that it might be easier to obtain an average bound of $\zeta(\frac{1}{2}+it)$ rather than a pointwise bound. In 1918, Hardy and Littlewood [22] proved that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 dt \sim T(\log T) \tag{1.2}$$

and in 1926 Ingham [29] proved that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt \sim \frac{T}{2\pi}(\log T)^4. \tag{1.3}$$

To date these are the only asymptotic results established for $I_k(T)$. In 1996, Conrey and Ghosh [10] conjectured that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^6 dt \sim \frac{42a_3}{9!} T(\log T)^9 \tag{1.4}$$

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and in 1998, Conrey and Gonek [11] conjectured that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^8 dt \sim \frac{24024a_4}{16!} T(\log T)^{16} \tag{1.5}$$

for certain specific constants a_3 and a_4 (see (1.8) below). In 1998, Keating and Snaith conjectured that

$$I_k(T) \sim \frac{g_k a_k}{(k^2)!} T(\log T)^{k^2} \tag{1.6}$$

where

$$g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!} \tag{1.7}$$

and

$$a_k = \prod_p \Big(1-\frac{1}{p}\Big)^{k^2} \sum_{m=0}^{\infty} \Big(\frac{\Gamma(m+k)}{m!\Gamma(k)}\Big)^2 p^{-m}. \tag{1.8}$$

Note that (1.6) agrees with (1.2), (1.3),(1.4), and (1.5).

The formulae (1.2) and (1.3) have been refined to asymptotic formulae with error terms admitting power savings. In 1926, Ingham [29] showed that

$$I_1(T) = T\mathcal{P}_2(\log T) + O(T^{\frac{1}{2}+\varepsilon}), \text{ for any } \varepsilon > 0, \tag{1.9}$$

and in 1979, Heath-Brown [24] showed that

$$I_2(T) = T\mathcal{P}_4(\log T) + O(T^{\frac{7}{8}+\varepsilon}), \text{ for any } \varepsilon > 0, \tag{1.10}$$

where $\mathcal{P}_2, \mathcal{P}_4$ are polynomials of degrees 2 and 4 respectively. The error term in (1.9) has been improved numerous times and the current record is due to Watt [46] who showed $O(T^{\frac{131}{416}+\varepsilon})$. The best error term for (1.10) is $O(T^{\frac{2}{3}+\varepsilon})$ due to Motohashi.

Although the asymptotic (1.6) remains open for $k \geq 3$, there are a number of results providing upper and lower bounds. Ramachandra established that $I_k(T) \gg_k T(\log T)^{k^2}$ for positive integers $2k$. This was extended to rational k by Heath-Brown and to all real $k \geq 0$ by Ramachandra, assuming the Riemann hypothesis. In 2008, Soundararajan [43] showed that on the Riemann hypothesis that $I_k(T) \ll_k T(\log T)^{k^2+\varepsilon}$, for any $\varepsilon > 0$. Building on this work and introducing a number of new ideas, Harper [23] showed that the Riemann hypothesis implies $I_k(T) \ll_k T(\log T)^{k^2}$.

In Ingham's article [29] on mean values of the Riemann zeta function, he studied the shifted the mean values

$$I_{\{a_1\},\{b_1\};\omega}(T) = \int_{-\infty}^{\infty} \zeta(\tfrac{1}{2}+a_1+it)\zeta(\tfrac{1}{2}+b_1-it)\omega(t)dt, \tag{1.11}$$

where a_1, b_1 are complex numbers satisfying $|a_1|, |b_1| \ll (\log T)^{-1}$ and $\omega(t) = \mathbb{1}_{[0,T]}(t)$ [1]. He showed that

$$I_{\{a_1\},\{b_1\};\omega}(T) = \int_{-\infty}^{\infty} \Big(\zeta(1+a_1+b_1) + \Big(\frac{t}{2\pi}\Big)^{-a_1-b_1} \zeta(1-a_1-b_1)\Big)\omega(t)dt + O(T^{\frac{1}{2}-\Re(a_1+b_1)/2}\log T), \tag{1.12}$$

where there error term is uniform in a_1 and b_1. His result for $I_1(T)$ may be derived by letting $a_1, b_1 \to 0$ (Observe that the integrand is entire in a_1 and b_1 since the poles cancel). In [34, Theorem 4.2, pp.171-178],

[1] $\mathbb{1}_B(t)$ is the indicator function of $B \subset \mathbb{R}$.

Motohashi used spectral theory to develop an asymptotic formula for

$$I_{\{a_1,a_2\},\{b_1,b_2\};\omega}(T) = \int_{-\infty}^{\infty} \zeta(\tfrac{1}{2}+a_1+it)\zeta(\tfrac{1}{2}+a_2+it)\zeta(\tfrac{1}{2}+b_1-it)\zeta(\tfrac{1}{2}+b_2-it).\omega(t)dt. \tag{1.13}$$

Let

$$Z(x_1,x_2,y_1,y_2) = \frac{\zeta(1+x_1+y_1)\zeta(1+x_1+y_2)\zeta(1+x_2+y_1)\zeta(1+x_2+y_2)}{\zeta(2+x_1+x_2+y_1+y_2)}.$$

Motohashi's theorem implies that

$$\begin{aligned} I_{\{a_1,a_2\},\{b_1,b_2\};\omega}(T) \sim \int_{-\infty}^{\infty} \Big(& Z(a_1,a_2,b_1,b_2) \\ & + (\tfrac{t}{2\pi})^{-a_1-b_1} Z(-b_1,a_2,-a_1,b_2) + (\tfrac{t}{2\pi})^{-a_1-b_2} Z(-b_2,a_2,b_1,-a_1) \\ & + (\tfrac{t}{2\pi})^{-a_1-b_1} Z(a_1,-b_1,-a_2,b_2) + (\tfrac{t}{2\pi})^{-a_2-b_2} Z(a_1,-b_2,b_1,-a_2) \\ & + (\tfrac{t}{2\pi})^{-a_1-a_2-b_1-b_2} Z(-b_1,-b_2,-a_1,-a_2)\Big)\omega(t)dt. \end{aligned} \tag{1.14}$$

This form of his theorem was observed in [8, p. 52, eq. (1.7.12)]. In fact, Motahashi's result is much more precise and he gives an exact formula for (1.13). Based on (1.12) and (1.14), it would be desirable to have a generalization of these formulae for shifted moments of zeta with more than four shifts. Inspired by (1.14), Conrey et al. [8] developed a conjecture for shifted moments of the Riemann zeta function. This shall be described shortly. They considered the mean values

$$I_{\mathcal{I},\mathcal{J};\omega}(T) = \int_{-\infty}^{\infty} \Big(\prod_{j=1}^{k} \zeta(\tfrac{1}{2}+a_j+it)\zeta(\tfrac{1}{2}+b_j-it)\Big)\omega(t)dt \tag{1.15}$$

where ω is a suitable smooth function, $\mathcal{I} = \{a_1,\ldots,a_k\}$ and $\mathcal{J} = \{b_1,\ldots,b_k\}$. We now explain the conjecture of [8] for this mean value, but we shall follow the notation of [28]. In order to do this, we shall first define several functions.

Definition 1. *Let X be a finite multiset of complex numbers. We define the arithmetic function $\sigma_X(n)$ to be the coefficient of n^{-s} in the Dirichlet series $\zeta_X(s)$, defined by $\zeta_X(s) := \prod_{x\in X}\zeta(s+x)$. In other words, if $X = \{x_1,\ldots,x_k\}$ then $\sigma_X(n) = \sum_{n_1\cdots n_k=n} n_1^{-x_1}\cdots n_k^{-x_k}$.*

Observe that if $X = \{0,\ldots,0\}$, $\zeta_X(s) = \zeta(s)^k$ where $k = \#X$. Thus if, the elements of X are close to zero, then $\zeta_X(s)$ may be thought of as a shifted version of $\zeta^k(s)$, where $k = \#X$.

Definition 2. *Given finite multisets X, Y of complex numbers we define the Dirichlet series.*

$$\mathcal{Z}_{X,Y}(s) := \sum_{n=1}^{\infty} \frac{\sigma_X(n)\sigma_Y(n)}{n^{1+s}}.$$

The series $\mathcal{Z}_{X,Y}(s)$ plays an important role in the study of $I_{\mathcal{I},\mathcal{J};\omega}(T)$ and will occur frequently in this article. It should be noted that $\mathcal{Z}_{X,Y}(s)$ has an analytic continuation to the left of $\Re(s) = 0$. In fact,

$$\mathcal{Z}_{X,Y}(s) = \Big(\prod_{x\in X, y\in Y}\zeta(1+s+x+y)\Big)\mathcal{A}_{X,Y}(s)$$

where $\mathcal{A}_{X,Y}(s)$ is holomorphic in a half-plane containing $s=0$. Precise formulae for $\mathcal{A}_{X,Y}(s)$ are given in Lemma 4 which follows.

Examples. (i) Let $X = \{x_1\}$ and $Y = \{y_1\}$. Then $\mathcal{Z}_{X,Y}(s) = \zeta(1+s+x_1+y_1)$ and $\mathcal{Z}_{\{a_1\},\{b_1\}}(0) = \zeta(1+a_1+b_1)$. Note that the integrand of (1.12) may be rewritten as

$$\Big(\mathcal{Z}_{\{a_1\},\{b_1\}}(0) + \Big(\frac{t}{2\pi}\Big)^{-a_1-b_1} \mathcal{Z}_{\{-b_1\},\{-a_1\}}(0)\Big)\omega(t).$$

(ii) Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. Then a calculation, using a formula of Ramanujan, establishes that

$$\mathcal{Z}_{X,Y}(s) = \frac{\zeta(1+s+x_1+y_1)\zeta(1+s+x_1+y_2)\zeta(1+s+x_2+y_1)\zeta(1+s+x_2+y_2)}{\zeta(2+2s+x_1+x_2+y_1+y_2)}.$$

Observe that $\mathcal{Z}_{\{a_1,a_2\},\{b_1,b_2\}}(0) = Z(a_1, a_2, b_1, b_2)$ and that integrand in (1.14) involves $Z_{X,Y}(0)$ for various sets X and Y of size 2 with entries chosen from $\{\pm a_j, \pm b_j\}$ for $j = 1, 2$. Based on these two examples, it is reasonable to expect that the general case of $|X| = |Y| = k$ is similar. In order to formulate the conjecture on the size of $I_{\mathcal{I},\mathcal{J};\omega}(T)$, we require a definition.

Definition 3. *Let $\mathcal{I} = \{a_1, \ldots, a_k\}$ and let $\mathcal{J} = \{b_1, \ldots, b_k\}$. Let Φ_j be the subsets of $\mathcal{I}$ of cardinality j and let Ψ_j be the subsets of $\mathcal{J}$ of cardinality j for $0 \le j \le k$. If $\mathcal{S} \in \Phi_j$ and $\mathcal{T} \in \Psi_j$ then write $\mathcal{S} = \{a_{i_1}, \ldots, a_{i_j}\}$ and $\mathcal{T} = \{b_{l_1}, \ldots, b_{l_j}\}$ where $i_1 < \cdots < i_j$ and $l_1 < \cdots < l_j$. We define $(\mathcal{I}_{\mathcal{S}}; \mathcal{J}_{\mathcal{T}})$ be the 2k-tuple obtained from $(a_1, a_2, \ldots, a_k; b_1, b_2, \ldots, b_k)$ by replacing a_{i_r} with $-b_{i_r}$ and b_{i_r} with $-a_{i_r}$ for $1 \le r \le j$.*

In order to explain this we give some simple examples.

Examples. Let $\mathcal{I} = \{a_1, a_2, a_3\}$, $\mathcal{J} = \{b_1, b_2, b_3\}$. If $\mathcal{S} = \emptyset$ and $\mathcal{T} = \emptyset$, then $(\mathcal{I}_{\mathcal{S}}; \mathcal{J}_{\mathcal{T}}) = (\mathcal{I}; \mathcal{J})$. If $\mathcal{S} = \{a_1\}$ and $\mathcal{T} = \{b_3\}$, then $(\mathcal{I}_{\mathcal{S}}; \mathcal{J}_{\mathcal{T}}) = (-b_3, a_2, a_3; b_1, b_2, -a_1)$. If $\mathcal{S} = \{a_1, a_3\}$ and $\mathcal{T} = \{b_2, b_3\}$, then $(\mathcal{I}_{\mathcal{S}}; \mathcal{J}_{\mathcal{T}}) = \{a_2, -b_2, -b_3; b_1, -a_1, -a_3\}$. If $\mathcal{S} = \mathcal{I}$ and $\mathcal{T} = \mathcal{J}$, then $(\mathcal{I}_{\mathcal{S}}; \mathcal{J}_{\mathcal{T}}) = (-\mathcal{J}; -\mathcal{I})$. The cases of $|\mathcal{S}| = |\mathcal{T}| = 0$ are called 0-swaps, the cases of $|\mathcal{S}| = |\mathcal{T}| = 1$ are called 1-swaps, and in general the cases of $|\mathcal{S}| = |\mathcal{T}| = k$ are called k-swaps. This terminology is introduced in the series of articles [13], [14], [15], and [16].

We are now prepared to state the conjecture of Conrey, Farmer, Keating, Rubinstein, and Snaith [8] for $I_{\mathcal{I},\mathcal{J};\omega}(T)$.

Conjecture 1. *Let $T > 3$. Let $\mathcal{I} = \{a_1, \ldots, a_k\}$, let $\mathcal{J} = \{b_1, \ldots, b_k\}$, and assume that $|a_i|, |b_j| \ll (\log T)^{-1}$. Let Φ_j be the subsets of $\mathcal{I} = \{a_1, \ldots, a_k\}$ of cardinality j and let Ψ_j be the subsets of $\mathcal{J} = \{b_1, \ldots, b_k\}$ of cardinality j for $0 \le j \le k$. Then for T sufficiently large*

$$I_{\mathcal{I},\mathcal{J};\omega}(T) = \int_{-\infty}^{\infty} \Big(\sum_{j=0}^{k} \sum_{\substack{\mathcal{S}\in\Phi_j \\ \mathcal{T}\in\Psi_j}} \mathcal{Z}_{\mathcal{I}_{\mathcal{S}},\mathcal{J}_{\mathcal{T}}}(0) \Big(\frac{t}{2\pi}\Big)^{-\mathcal{S}-\mathcal{T}} + o(1) \Big)\omega(t)dt \tag{1.16}$$

where ω is a nice weight function, and where we have defined

$$\Big(\frac{t}{2\pi}\Big)^{-\mathcal{S}-\mathcal{T}} := (t/2\pi)^{-\sum_{x\in\mathcal{S}} x - \sum_{y\in\mathcal{T}} y} \tag{1.17}$$

for $\mathcal{S} \in \Phi_j$ and $\mathcal{T} \in \Psi_j$.

Remarks

(1) The works of Ingham and Motohashi establish this conjecture in the cases $|\mathcal{I}| = |\mathcal{J}| = 1$ and $|\mathcal{I}| = |\mathcal{J}| = 2$.

(2) Conrey, Farmer, Keating, Rubinstein, and Snaith [8] made this conjecture with the $o(1)$ replaced by $O(T^{-\frac{1}{2}+\varepsilon})$. That is, the total error with the weight included is $O(T^{\frac{1}{2}+\varepsilon})$. They gave a heuristic argument based on a "recipe" (see [8, section 2.2, pp. 53-56]).

(3) There is some debate on the size of the error term in this conjecture. In the case $k = 3$ and all shifts $a_i = b_j = 0$, namely $I_3(T)$, Motohashi [34, p.218, eq. (5.4.10)] has conjectured that the error term is $\Omega(T^{\frac{3}{4}-\delta})$ for any fixed $\delta > 0$. Similarly, Ivic [30, p. 171] has conjectured that the error term in this case is $O(T^{\frac{3}{4}+\varepsilon})$ and $\Omega(T^{\frac{3}{4}})$.

(4) Zhang [47] studied a related mean value (the cubic moment of quadratic Dirichlet L-functions at the central point) and found a main term plus secondary term of size $T^{\frac{3}{4}}$.

(5) Recent numerical calculations of these moments currently do not seem give to conclusive evidence of what is the correct size for the error term.

In this article we shall prove that a certain ternary additive divisor bound implies an asymptotic formula for $I_{\mathfrak{I},\mathfrak{J};\omega}(T)$ in the case $|\mathfrak{I}| = |\mathfrak{J}| = 3$. In the remainder of this article we consider $\mathfrak{I}, \mathfrak{J}$ where

$$\mathfrak{I} = \{a_1, a_2, a_3\} \text{ and } \mathfrak{J} = \{b_1, b_2, b_3\} \tag{1.18}$$

consist of complex numbers with the size restriction

$$|a_i|, |b_j| \ll \frac{1}{\log T}. \tag{1.19}$$

The family of additive divisor sums we are concerned with are

$$D_{f;\mathfrak{I},\mathfrak{J}}(r) = \sum_{m-n=r} \sigma_{\mathfrak{I}}(m)\sigma_{\mathfrak{J}}(n)f(m,n) \tag{1.20}$$

$r \in \mathbb{Z} \setminus \{0\}$ and f is a smooth function. Moreover, the partial derivatives of f satisfy growth conditions. That is, there exist X, Y, and P positive such that

$$\text{support}(f) \subset [X, 2X] \times [Y, 2Y] \tag{1.21}$$

and the partial derivatives satisfy

$$x^i y^j f^{(i,j)}(x,y) \ll P^{i+j}. \tag{1.22}$$

In order to state a conjecture for the size of $D_{f;\mathfrak{I},\mathfrak{J}}(h)$, we must introduce several multiplicative functions.

Definition 4. *Let $X = \{x_1, \ldots, x_k\}$ be a finite multiset of complex numbers and $s \in \mathbb{C}$. The multiplicative function $n \to g_X(s,n)$ is given by*

$$g_X(s,n) = \prod_{p^\alpha || n} \frac{\sum_{j=0}^{\infty} \frac{\sigma_X(p^{j+\alpha})}{p^{js}}}{\sum_{j=0}^{\infty} \frac{\sigma_X(p^j)}{p^{js}}}. \tag{1.23}$$

In other words, for $n \in \mathbb{N}$ we have $\sum_{m=1}^{\infty} \frac{\sigma_X(nm)}{m^s} = g_X(s,n)\zeta(s+x_1)\cdots\zeta(s+x_k)$.

The multiplicative function $n \to G_X(s,n)$ is given by

$$G_X(s,n) = \sum_{d|n} \frac{\mu(d)d^s}{\phi(d)} \sum_{e|d} \frac{\mu(e)}{e^s} g_X\Big(s, \frac{ne}{d}\Big). \tag{1.24}$$

With these definitions in hand, we may state the additive divisor conjecture.

Conjecture 2 (Additive divisor conjecture $\mathcal{AD}(\vartheta, C)$)**.** *Let $X, Y, P \geq 1$ be positive parameters such that $Y \asymp X$, and f is a smooth function satisfying* (1.21) *and* (1.22)*. Let $\mathfrak{I} = \{a_1, a_2, a_3\}$ and $\mathfrak{J} = \{b_1, b_2, b_3\}$ be sets of distinct complex numbers satisfying $|a_i|, |b_j| \ll (\log X)^{-1}$. Then there exist (ϑ, C) where $\vartheta \in [\frac{1}{2}, \frac{2}{3})$*

and $C>0$ such that for every $\varepsilon_1,\varepsilon_2>0$

$$D_{f;\mathfrak{I},\mathfrak{J}}(r)=\sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\prod_{j_1\neq i_1}\zeta(1-a_{i_1}+a_{j_1})\prod_{j_2\neq i_2}\zeta(1-b_{i_2}+b_{j_2})\cdot$$
$$\sum_{q=1}^{\infty}\frac{c_q(r)G_{\mathfrak{I}}(1-a_{i_1},q)G_{\mathfrak{J}}(1-b_{i_2},q)}{q^{2-a_{i_1}-b_{i_2}}}\int_{\max(0,r)}^{\infty}f(x,x-r)x^{-a_{i_1}}(x-r)^{-b_{i_2}}\,dx+O(P^CX^{\vartheta+\varepsilon_1}),$$

uniformly for $1\le r\le X^{\frac12+\varepsilon_2}$.

Remarks.

(1) The main term in the above conjecture can be derived by following Duke, Friedlander, and Iwaniec's δ-method [18].

(2) The leading term in the conjecture for $\sum_{n\le x}d_k(n)d_k(n+h)$ can be worked out with a heuristic probabilistic calculation. This was recently done independently by Tao [44] and Ng and Thom [36].

(3) In the case that $\sigma_{\mathfrak{I}}(n)=\sigma_{\mathfrak{J}}(n)=d(n)$, the divisor function, Duke, Friedlander, and Iwaniec [18] have shown that an analogous result is available with an error term having $\vartheta=\frac34$ and $C=\frac54$. Furthermore, they mention that improvements to their argument would reduce C to $\frac34$ and more elaborate arguments may lead to $C=\frac12$.

(4) Conrey and Gonek [11] have conjectured that in the case of $D_3(x,r)$ (the unsmoothed version $D_{f;\mathfrak{I},\mathfrak{J}}(r)$) that $\vartheta=\frac12$ is valid for $1\le r\le\sqrt{x}$. Moreover, Conrey and Keating [15] have suggested that $\vartheta=\frac12$ is valid for $1\le r\le x^{1-\varepsilon}$. This is discussed extensively in Ng and Thom [36] where a probabilstic argument has been given which suggests the error term for $D_k(x,r)$ is uniform in the range $1\le r\le x^{1-\varepsilon}$. Hence it is likely that the above conjecture holds in the wider range $r\le X^{1-\varepsilon_2}$.

(5) Blomer [5] has shown that there exists $C>0$ such that

$$\sum_{\ell_1m-\ell_2n=h}a(m)\overline{a(n)}f(m,n)\ll P^CX^{\frac12+\Theta+\varepsilon}$$

where $g(z)=\sum_{m=1}^{\infty}a(m)m^{\frac{k-1}{2}}e(mz)\in S_k(N,\chi)$ is a primitive cusp form (holomorphic newform) and Θ is a non-negative constant such that $|\lambda(n)|\ll n^{\Theta}$ for for eigenvalues $\lambda(n)$ of the Hecke operator T_n acting on the space of weight 0 Maass cusp forms of level N.

(6) Recently, Aryan [2] has shown in the case that $\sigma_{\mathfrak{I}}(n)=\sigma_{\mathfrak{J}}(n)=d(n)$, $X=Y$, and $P=1$, that the corresponding error term is $O(X^{\frac12+\Theta+\varepsilon})$.

(7) Unfortunately, for $k\ge3$ this currently remains open. In the case of the unsmoothed sum $D_k(x,r)$ uniform upper and lower bounds for $r\le x^A$, for $A>0$, of the correct order of magnitude are known. Ng and Thom [36] have established lower bounds and Daniel [17] and Henriot [27] have established upper bounds.

We now introduce a convenient weight ω. Let ω satisfy the following:

(1.25) ω is smooth,

(1.26) the support of ω lies in $[c_1T,c_2T]$ where $0<c_1<c_2$,

(1.27) there exists $T_0>0$ such that for every $\varepsilon>0$, $T^{\frac34+\varepsilon}\le T_0\ll T$ and $\omega^{(j)}(t)\ll T_0^{-j}$.

The main goal of this article is to show that Conjecture 2 implies Conjecture 1.

Theorem 1.1. *Let $\mathcal{I} = \{a_1, a_2, a_3\}$, $\mathcal{J} = \{b_1, b_2, b_3\}$, and assume the elements of $\mathcal{I}$ and $\mathcal{J}$ satisy* (1.19)*. Assume Conjecture 2 holds for some positive θ and C, then for any $\varepsilon > 0$*

$$\begin{aligned} I_{\mathcal{I},\mathcal{J};\omega}(T) &= \int_{-\infty}^{\infty} \Big(\sum_{j=0}^{3} \sum_{\substack{\mathcal{S}\in\Phi_j \\ \mathcal{T}\in\Psi_j}} \mathcal{Z}_{\mathcal{I}_{\mathcal{S}},\mathcal{J}_{\mathcal{T}}}(0) \Big(\frac{t}{2\pi}\Big)^{-\mathcal{S}-\mathcal{T}} \Big) \omega(t) dt \\ &\quad + O\Big(T^{\frac{3\vartheta}{2}+\varepsilon} \Big(\frac{T}{T_0}\Big)^{1+C} + \eta_C T^{\frac{3}{2}C - \frac{1}{2} + \frac{3\vartheta}{2} + \varepsilon} T_0^{-C} + T^{\frac{1}{2}+\varepsilon} \Big) \end{aligned} \tag{1.28}$$

where $\eta_C = 1$ if $C \geq 1$ and $\eta_C = 0$ if $0 < C < 1$.

From this theorem, we deduce an asymptotic formula with power savings error term for the sixth moment of the Riemann zeta function.

Corollary 1.2. *If Conjecture 2 ($\mathcal{AD}(\vartheta, C)$) is true with $\vartheta \in [\frac{1}{2}, \frac{2}{3})$ and $C \in (0,1)$, then there exists a polynomial $\mathcal{P}_9(x)$ of degree 9 such that*

$$I_3(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^6 dt = T\mathcal{P}_9(\log T) + O(T^{\frac{\frac{3\vartheta}{2}+1+C}{2+C}+\varepsilon}) \tag{1.29}$$

and in particular (1.4) *holds. If Conjecture 2 ($\mathcal{AD}(\vartheta, C)$) is true with $\vartheta \in [\frac{1}{2}, \frac{2}{3})$ and $1 \leq C < 3(1-\vartheta)$, then for any $\varepsilon > 0$*

$$I_3(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^6 dt = T\mathcal{P}_9(\log T) + O(T^{1-\varepsilon}). \tag{1.30}$$

Remarks.

(1) Conditionally, this confirms Conjecture 1.
(2) This result makes rigorous the argument in [11]. In their work they argued that the $I_3(T)$ is asymptotic to the sum of mean values of the shape $\int_T^{2T} |\mathbb{D}_{T^{\theta_i}}(\frac{1}{2}+it)|^2$ dt where $\mathbb{D}_{T^{\theta_i}}(s) = \sum_{n\leq T^{\theta_i}} d_3(n)n^{-s}$ and $\theta_1 + \theta_2 = 3$. They then invoked a Theorem of Goldston and Gonek [19] to asymptotically evaluate these expressions. This required certain conjectural formula for $D_3(x, r) = \sum_{n\leq x} d_3(n)d_3(n+r)$ with sharp error terms, uniform for $r \leq \sqrt{x}$.
(3) In a sense, this improves work of Ivic, who showed that certain asymptotic formula for $D_3(x, r)$ implies $I_3(T) \ll T^{1+\varepsilon}$ for any $\varepsilon > 0$. A slight difference in our treatment is that we have chosen to deal with the additive divisor sums corresponding to $\sigma_{\mathcal{I}}(n)$ and $\sigma_{\mathcal{J}}(n)$ where the elements of $\mathcal{I}$ and $\mathcal{J}$ are $\ll (\log T)^{-1}$. This is a mild assumption and it is likely than any proof leading to an asymptotic formula for $D_3(x, r)$ will also provide an asymptotic formula for $D_{f;\mathcal{I},\mathcal{J}}(r)$.
(4) In our proof we follow an argument of Hughes and Young [28] who evaluated twisted fourth moment
$$\int_{-\infty}^{\infty} \Big(\frac{h}{k}\Big)^{-it} |\zeta(\tfrac{1}{2} + it)|^4 \omega(t) dt.$$
for coprime natural numbers h, k satisfying $hk \leq T^{\frac{1}{11}-\varepsilon}$. Recently, this was improved by Bettin, Bui, Li, and Radziwiłł [4] to $hk \leq T^{\frac{1}{4}-\varepsilon}$.
(5) In the case that the additive divisor conjecture (Conjecture 2 ($\mathcal{AD}(\vartheta, C)$) is true with the best possible exponent $\vartheta = \frac{1}{2}$ and also $C < 1$ and $T_0 = T^{1-\varepsilon}$, this result shows that error term for $I_{\mathcal{I},\mathcal{J};\omega}(T)$ is $O(T^{\frac{3}{4}+\varepsilon})$. Note this matches with Ivic and Motohashi's speculations on the error term for $I_3(T)$.
(6) Formulae and numerical values for the coefficients of $\mathcal{P}_9$ may be found in [8], [9].

(7) From Theorem 1.1 we can also deduce formulae for the integrals

$$\int_{-\infty}^{\infty} \zeta^{(j_1)}(\tfrac{1}{2}+it)\zeta^{(j_2)}(\tfrac{1}{2}+it)\zeta^{(j_3)}(\tfrac{1}{2}+it)\zeta^{(j_4)}(\tfrac{1}{2}-it)\zeta^{(j_5)}(\tfrac{1}{2}-it)\zeta^{(j_6)}(\tfrac{1}{2}-it)\omega(t)dt,$$

assuming Conjecture 2. Such integrals can be used in detecting large gaps between the zeros of the Riemann zeta function. For instance see Hall [20].

Proof of 1.2. In this proof $\mathcal{I} = \{a_1, a_2, a_3\}$ and $\mathcal{J} = \{b_1, b_2, b_3\}$ are each triples of complex numbers. We also write $\vec{a} = (a_1, a_2, a_3)$, and $\vec{b} = (b_1, b_2, b_3)$. Set $f(\vec{a}; \vec{b}) = I_{\mathcal{I},\mathcal{J};\omega}(T)$ and

$$g(\vec{a};\vec{b}) = \int_{-\infty}^{\infty} \omega(t)\Big(\sum_{j=0}^{3} \sum_{\substack{\mathcal{S}\in\Phi_j \\ \mathcal{T}\in\Psi_j}} \mathcal{Z}_{\mathcal{I}_{\mathcal{S}},\mathcal{J}_{\mathcal{T}}}(0)\Big(\frac{t}{2\pi}\Big)^{-\mathcal{S}-\mathcal{T}}\Big)dt. \tag{1.31}$$

Note that $f(\vec{a};\vec{b})$ is holomorphic in a_i and b_j as long as $|a_i| < \frac{1}{2}$ and $|b_j| < \frac{1}{2}$. Also by Lemma 2.51 of [8] and [6, Sections 4.4,4.5] $g(\vec{a};\vec{b})$ is holomorphic in a_i and b_j as long as $|a_i| < \eta$ and $|b_j| < \eta$ for a sufficiently small fixed η. It shall be convenient to set $a_4 = -b_1, a_5 = -b_2$, and $a_6 = -b_3$. We have

$$g(\vec{a};\vec{b}) = \int_{-\infty}^{\infty} \omega(t)P(\log \tfrac{t}{2\pi}, \vec{a}, \vec{b})dt, \tag{1.32}$$

where

$$P(x,\vec{a},\vec{b}) = \frac{(-1)^3}{(3!)^2}\frac{1}{(2\pi i)^6}\oint\cdots\oint \frac{G(z_1,\dots,z_6)\Delta^2(z_1,\dots,z_6)}{\prod_{j=1}^{6}\prod_{i=1}^{6}(z_j-a_i)}e^{(x/2)\sum_{j=1}^{6}(z_j-z_{3+j})}dz_1\dots dz_6, \tag{1.33}$$

such that the integrals $\oint$ are over small, positively oriented circles centered at the a_i,

$$\Delta(z_1,\dots,z_6) = \prod_{1\le i<j\le 6}(z_j - z_i),$$

$$G(z_1,\dots,z_6) = A(z_1,\dots,z_6)\prod_{i=1}^{3}\prod_{j=1}^{3}\zeta(1+z_i-z_{3+j}),$$

$$A(z_1,\dots,z_6) = \prod_p\prod_{i=1}^{3}\prod_{j=1}^{3}\Big(1-\frac{1}{p^{1+z_i-z_{3+j}}}\Big)\int_0^1\prod_{j=1}^{3}\Big(1-\frac{e(\theta)}{p^{\frac{1}{2}+z_j}}\Big)^{-1}\Big(1-\frac{e(\theta)}{p^{\frac{1}{2}-z_{3+j}}}\Big)^{-1}.$$

It follows that for $|a_i|, |b_j| < \eta$ that $F(\vec{a};\vec{b}) = f(\vec{a};\vec{b}) - g(\vec{a};\vec{b})$ is holomorphic in each of the variables. Therefore by six applications of the maximum modulus principle

$$|F(\vec{0};\vec{0})| \le \mathrm{Max}_{a_i\in C_i, b_j\in\tilde{C}_j}|F(\vec{a};\vec{b})| \tag{1.34}$$

where $\vec{0} = (0,0,0)$, $C_i = \{z_i \in \mathbb{C} \mid |z_i| = r_i\}$, and $\tilde{C}_i = \{z_i \in \mathbb{C} \mid |z_i| = \rho_i\}$ for $i = 1,2,3$ where $|r_i|, |\rho_j| \ll (\log T)^{-1}$. It follows from Theorem 1.1 that

$$|F(\vec{0};\vec{0})| \ll T^{\frac{3\vartheta}{2}+\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C} + \eta_C T^{\frac{3}{2}C-\frac{1}{2}+\frac{3\vartheta}{2}+\varepsilon}T_0^{-C} + T^{\frac{1}{2}+\varepsilon}$$

and thus

$$\begin{aligned}\int_{-\infty}^{\infty}|\zeta(\tfrac{1}{2}+it)|^6\omega(t)dt &= \int_{-\infty}^{\infty}P_9\Big(\log\frac{t}{2\pi}\Big)\omega(t)dt \\ &\quad + O\Big(T^{\frac{3\vartheta}{2}+\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C} + \eta_C T^{\frac{3}{2}C-\frac{1}{2}+\frac{3\vartheta}{2}+\varepsilon}T_0^{-C} + T^{\frac{1}{2}+\varepsilon}\Big)\end{aligned} \tag{1.35}$$

where

$$P_9(x) = \frac{(-1)^3}{(3!)^2}\frac{1}{(2\pi i)^6}\oint\cdots\oint \frac{G(z_1,\ldots,z_6)\Delta^2(z_1,\ldots,z_6)}{\prod_{j=1}^6 z_j^6} e^{(x/2)\sum_{j=1}^6 (z_j - z_{3+j})} dz_1 \ldots dz_6, \tag{1.36}$$

and ω satisfies (1.25), (1.26), and (1.27). Now choose $\omega^+(t)$ to be a smooth majorant of the $\mathbb{1}_{[T,2T]}(t)$ with $\omega^+ = 1$ in $[T, 2T]$ and $\omega^+(t) = 0$ for $t < T - T_0$ and $2T + T_0$ and satisfying $(\omega^+)^{(j)} \ll T_0^{-j}$. It follows that

(1.37)

$$I_3(2T) - I_3(T) \le \int_T^{2T} P_9\Big(\log\frac{t}{2\pi}\Big)dt + O(T_0(\log T)^9 + T^{\frac{3\vartheta}{2}+\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C} + \eta_C T^{\frac{3}{2}C-\frac{1}{2}+\frac{3\vartheta}{2}+\varepsilon}T_0^{-C} + T^{\frac{1}{2}+\varepsilon}\Big).$$

The term $O(T_0(\log T)^9)$ arises from estimating the portions of the integral corresponding to the intervals $[T - T_0, T]$ and $[2T, 2T + T_0]$. In the case $C \le 1$ the third error term is dominated by the second. Now choose T_0 so that the first and second error terms are equal. Solving for T_0 we find that $T_0 = T^{\frac{\frac{3\vartheta}{2}+1+C}{2+C}}$ and thus

$$I_3(2T) - I_3(T) \le \int_T^{2T} P_9\Big(\log\frac{t}{2\pi}\Big)dt + O\Big(T^{\frac{\frac{3\vartheta}{2}+1+C}{2+C}+\varepsilon}\Big). \tag{1.38}$$

A similar argument with a smooth minorant $\omega^-(t)$ of $\mathbb{1}_{[T,2T]}(t)$ establishes the same lower bound and thus

$$I_3(2T) - I_3(T) = \int_T^{2T} P_9\Big(\log\frac{t}{2\pi}\Big)dt + O\Big(T^{\frac{\frac{3\vartheta}{2}+1+C}{2+C}+\varepsilon}\Big). \tag{1.39}$$

Substituting $\frac{T}{2^j}$ with $j = 1, 2, \ldots$, we find

$$I_3(T) = \int_0^T P_9\Big(\log\frac{t}{2\pi}\Big)dt + O\Big(T^{\frac{\frac{3\vartheta}{2}+1+C}{2+C}+\varepsilon}\Big) = T\mathcal{P}_9(\log T) + O\Big(T^{\frac{\frac{3\vartheta}{2}+1+C}{2+C}+\varepsilon}\Big) \tag{1.40}$$

for some polynomial $\mathcal{P}_9$. In the case that $C > 1$ the third error term in (1.37) is now present. We apply a similar argument to as before, however we choose $T_0 = T^{1-\varepsilon}$. Thus the error term in (1.28) is $O((T^{\frac{3\vartheta}{2}} + T^{\frac{C-1}{2}+\frac{3\vartheta}{2}})T^{\varepsilon})$. In order for this to be $O(T^{1-\varepsilon})$ we require that $\vartheta < \frac{2}{3}$ and $\frac{C-1}{2} + \frac{3\vartheta}{2} < 1$. The second condition is exactly $\vartheta + \frac{C}{3} < 1$. Following the same argument as above we arrive at (1.30). ☐

1.1. **Conventions and Notation.** In this article we shall use the convention that ε denotes an arbitrarily small positive constant which may vary from line to line. Given two functions $f(x)$ and $g(x)$, we shall interchangeably use the notation $f(x) = O(g(x))$, $f(x) \ll g(x)$, and $g(x) \gg f(x)$ to mean there exists $M > 0$ such that $|f(x)| \le M|g(x)|$ for all sufficiently large x. We write $f(x) \asymp g(x)$ to mean that the estimates $f(x) \ll g(x)$ and $g(x) \ll f(x)$ simultaneously hold. If we write $f(x) = O_a(g(x))$, $f(x) \ll_a g(x)$, or $f(x) \asymp_a g(x)$, then we mean that the corresponding constants depend on a. The letter p will always be used to denote a prime number. For a function $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$, $\varphi^{(m,n)}(x, y) = \frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial y^n}\varphi(x, y)$. Throughout this article we often use the fact that $t \in [c_1T, c_2T]$ so that $t \asymp T$.

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2. The approximate functional equation and the Dirichlet series $\mathcal{Z}_{\mathcal{I},\mathcal{J}}(s)$

One of the difficulties in evaluating mean values of the type (1.1) and (1.15) is that the integration is on the line $\Re(s) = 1/2$ where ζ does not possess an absolutely convergent Dirichlet series. Instead, in the critical strip a standard tool is the approximate functional equation. The approximate functional equation for $\zeta(s)^k$ for $k=1,2$ were derived by Hardy and Littlewood. Their result asserts that

$$\zeta(s)^k = \sum_{n\le x} \frac{d_k(n)}{n^s} + \chi(s)^k \sum_{n\le y} \frac{d_k(n)}{n^{1-s}} + \mathrm{error}(s,x,y) \tag{2.1}$$

where $xy \asymp (\frac{t}{2\pi})^k$. There are several problems with this version of the approximate functional equation. First, each of these sums have sharp cutoffs, that is, the sum over n does not decay smoothly. In practice, it is convenient to sum over all integers with a weight which is smooth. The sharp cutoff functions lead to poor error terms $\mathrm{error}(s,x,y)$. Another problem is the presence of the factor $\chi(s)^k$. More modern versions of the approximate functional (see [45, p.92, eq. (4.20.1)]) equation have the shape

$$\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \nu_t(n,x) + \chi(s)^k \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{1-s}} \tilde{\nu}_t(n,y) + O(\exp(-ct^2)) \tag{2.2}$$

for certain smooth weights $\nu_t(n,x)$ and $\tilde{\nu}_t(n,y)$ where $xy \asymp (\frac{t}{2\pi})^k$ and $s=\frac12+it$. A classical approach to evaluating (1.1) is to use the identity $|\zeta(\frac12+it)|^{2k} = \zeta(\frac12+it)^k\zeta(\frac12-it)^k$ and then to apply (2.2) with $s=1/2\pm it$ and then multiply out. Namely,

$$\begin{aligned} I_{\mathcal{I},\mathcal{J};\omega}(T) &= \int_{\mathbb{R}} \Big| \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{\frac12+it}} \nu_t(n,x) \Big|^2 \omega(t)dt + \int_{\mathbb{R}} \Big| \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{\frac12-it}} \tilde{\nu}_t(n,y) \Big|^2 dt \\ &\quad + \int_{\mathbb{R}} \chi(\tfrac12-it)^k \sum_{m,n=1}^{\infty} \frac{d_k(m)d_k(n)}{m^{\frac12+it} n^{\frac12+it}} \nu_t(m,x)\tilde{\nu}_t(n,y)dt + \cdots . \end{aligned} \tag{2.3}$$

The first two sums can be asymptotically estimated by standard mean values techniques if $x,y \ll T$. However, in the range $T \ll x,y \ll T^{2-\varepsilon}$, then these can only be evaluated if sharp asymptotic estimates for the correlation sums $D_k(x,r) = \sum_{n\le x} d_k(n)d_k(n+r)$. are available. An additional difficulty with (2.3) is the presence of the oscillating factor $\chi(\frac12-it)^k$. By Stirling's formula these lead to integrals of the shape $\int_{\mathbb{R}} \exp(it\log\frac{(t/2\pi e)^k}{mn})\nu_t(m,x)\tilde{\nu}_t(n,y)dt$. These have to be treated with stationary phase and they lead to unappealing oscillating factors. One way to circumvent the factors $\chi(\frac12-it)^k$ is to develop an approximate functional equation for $|\zeta(\frac12+it)|^{2k}$ instead of $\zeta(\frac12+it)^k$. This idea is due to Heath-Brown who showed that

$$|\zeta(\tfrac12+it)|^{2k} = 2\sum_{m,n=1}^{\infty} \frac{d_k(m)d_k(n)}{m^{\frac12+it}n^{\frac12-it}} W_t(mn) + O(e^{-t^2/2}) \text{ for } t\ge 1 \tag{2.4}$$

where $W_t(u)$ is a smooth weight function supported in $[1, ct^k]$ for some $c>0$.

In this section, we prove an approximate functional equation for $\zeta_{\mathcal{I}}(\frac12+it)\zeta_{\mathcal{J}}(\frac12-it)$ where $\mathcal{I}=\{a_1,a_2,a_3\}$ and $\mathcal{J}=\{b_1,b_2,b_3\}$ analogous to (2.4). Recall that $\zeta_{\mathcal{I}}$ and $\zeta_{\mathcal{J}}$ are defined in Definition 1. The following proposition is a straight forward generalization of [28, Proposition 2.1, p. 209] which handles the case $\mathcal{I}=\{a_1,a_2\}$ and $\mathcal{J}=\{b_1,b_2\}$. Before we state the proposition we must define a convenient polynomial which will be used in the proposition and in our main theorem.

Definition. Let $Q_{\mathcal{I},\mathcal{J}}(s)$ where $\mathcal{I}=\{a_1,a_2,a_3\}$ and $\mathcal{J}=\{b_1,b_2,b_3\}$ be an even polynomial satisfying the following properties: $Q_{\mathcal{I},\mathcal{J}}(0)=1$, $Q_{\mathcal{I},\mathcal{J}}(s)$ is symmetric in the a_i's, and b_i's, invariant under $a_i\to -a_i$ and $b_i \to -b_i$, and vanishes at $\frac12 - \frac{a_1+b_1}{2}$ and at other points obtained by the previous symmetries.

In our argument we shall express $I_{\mathcal{I},\mathcal{J};\omega}(T)$ as a certain multivariable integral. This integral shall be computed by moving contour integrals to the left. At one point some unwanted poles near $s=\frac12$ shall arise. The role of $Q_{\mathcal{I},\mathcal{J}}(s)$ is to cancel these poles and avoid extraneous terms.

Proposition 2.1. *Let $\mathcal{I} = \{a_1, a_2, a_3\}$ and $\mathcal{J} = \{b_1, b_2, b_3\}$. Let $G(s)$ be an even, entire function of rapid decay* [2] *as $|s| \to \infty$ in any fixed strip $|\Re(s)| \le A$ with $G(0) = 1$, and divisible by $Q_{\mathcal{I},\mathcal{J}}(s)$. Let*

$$V_{\mathcal{I},\mathcal{J};t}(x) = \frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s} g_{\mathcal{I},\mathcal{J}}(s,t)x^{-s}ds, \tag{2.5}$$

where

$$g_{\mathcal{I},\mathcal{J}}(s,t) = \prod_{i=1}^{3} \frac{\Gamma(\frac{\frac{1}{2}+a_i+s+it}{2})\Gamma(\frac{\frac{1}{2}+b_i+s-it}{2})}{\Gamma(\frac{\frac{1}{2}+a_i+it}{2})\Gamma(\frac{\frac{1}{2}+b_i-it}{2})}. \tag{2.6}$$

Furthermore, set

$$X_{\mathcal{I},\mathcal{J};t} = \pi^{\sum_{i=1}^{3} a_i+b_i} \prod_{i=1}^{3} \frac{\Gamma(\frac{\frac{1}{2}-a_i-it}{2})\Gamma(\frac{\frac{1}{2}-b_i+it}{2})}{\Gamma(\frac{\frac{1}{2}+a_i+it}{2})\Gamma(\frac{\frac{1}{2}+b_i-it}{2})}. \tag{2.7}$$

Then for any $A' > 0$, we have

(2.8)

$$\zeta_{\mathcal{I}}(\tfrac{1}{2}+it)\zeta_{\mathcal{J}}(\tfrac{1}{2}-it) = \sum_{m,n=1}^{\infty} \frac{\sigma_{\mathcal{I}}(m)\sigma_{\mathcal{J}}(n)}{(mn)^{\frac{1}{2}}}\Big(\frac{m}{n}\Big)^{-it} V_{\mathcal{I},\mathcal{J};t}(\pi^3 mn) + X_{\mathcal{I},\mathcal{J};t}\sum_{m,n=1}^{\infty} \frac{\sigma_{-\mathcal{J}}(m)\sigma_{-\mathcal{I}}(n)}{(mn)^{\frac{1}{2}}}\Big(\frac{m}{n}\Big)^{-it} V_{-\mathcal{J},-\mathcal{I};t}(\pi^3 mn)$$
$$+ O((1+|t|)^{-A'}).$$

Remark. This proposition can be generalized to the case $\mathcal{I} = \{a_1, \dots, a_k\}$ and $\mathcal{J} = \{b_1, \dots, b_k\}$.

Proof. Throughout this proof we let $\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ and we make use of the functional equation $\Lambda(s) = \Lambda(1-s)$. Set

$$\Lambda_{\mathcal{I},\mathcal{J}}(s) = \prod_{i=1}^{3} \Lambda(\tfrac{1}{2}+s+a_i+it)\Lambda(\tfrac{1}{2}+s+b_i-it), \tag{2.9}$$

and

$$I_1 = \frac{1}{2\pi i}\int_{(1)} \Lambda_{\mathcal{I},\mathcal{J}}(s)\frac{G(s)}{s}ds. \tag{2.10}$$

We shall move the contour to the left to the line $\Re(s) = -1$ and apply the residue theorem. The integrand has poles at $s = 0$ and at $s = \frac{1}{2} - a_i - it$ and $s = \frac{1}{2} - b_i - it$ with $i = 1, 2, 3$. The residue at $s = 0$ is

$$\Lambda_{\mathcal{I},\mathcal{J}}(0) = \prod_{i=1}^{3} \Lambda(\tfrac{1}{2}+a_i+it)\Lambda(\tfrac{1}{2}+b_i-it).$$

Each residue at the other poles is $O((1+|t|)^{-A})$ due to the rapid decrease of $G(s)$ when $|\Im(s)|$ is large. Let

$$I_2 = \frac{1}{2\pi i}\int_{(-1)} \Lambda_{\mathcal{I},\mathcal{J}}(s)\frac{G(s)}{s}ds.$$

By the residue theorem it follows that

$$I_1 - I_2 = \Lambda_{\mathcal{I},\mathcal{J}}(0) + O((1+|t|)^{-A}).$$

Now observe that

$$\Lambda_{I,\mathcal{J}}(-s) = \Lambda_{-\mathcal{J},-\mathcal{I}}(s)$$

[2] G is of rapid decay if for every $B > 0$, we have $|G(s)| \le |s|^{-B}$ for $|\Re(s)| \le A$ and $|\Im(s)|$ sufficiently large. An admissible G is $G(s) = Q_{\mathcal{I},\mathcal{J}}(s)\exp(s^2)$. Observe that A may be chosen to be any positive constant.

and thus

$$I_2 = -\frac{1}{2\pi i}\int_{(1)} \Lambda_{-\mathcal{J},-\mathcal{I}}(s)\frac{G(s)}{s}ds.$$

Set

$$\mathbf{Z}_{\mathcal{I},\mathcal{J},t}(s) = \prod_{i=1}^{3} \zeta(\tfrac{1}{2}+s+a_i+it)\zeta(\tfrac{1}{2}+s+b_i-it) \tag{2.11}$$

and

$$\mathbf{G}_{\mathcal{I},\mathcal{J},t}(s) = \pi^{-\frac{3}{2}-3s-\frac{1}{2}\sum_{j=1}^{3}(a_j+b_j)} \prod_{i=1}^{3} \Gamma(\tfrac{1}{2}+s+a_i+it)\Gamma(\tfrac{1}{2}+s+b_i-it). \tag{2.12}$$

A calculation using the definition (2.9) establishes that

$$\Lambda_{\mathcal{I},\mathcal{J}}(s) = \mathbf{Z}_{\mathcal{I},\mathcal{J},t}(s)\mathbf{G}_{\mathcal{I},\mathcal{J},t}(s). \tag{2.13}$$

It follows that

$$\begin{aligned}\mathbf{Z}_{\mathcal{I},\mathcal{J},t}(0) = {} & \frac{1}{2\pi i}\int_{(1)} \mathbf{Z}_{\mathcal{I},\mathcal{J},t}(s)\frac{\mathbf{G}_{\mathcal{I},\mathcal{J},t}(s)}{\mathbf{G}_{\mathcal{I},\mathcal{J},t}(0)}\frac{G(s)}{s}ds + \frac{1}{2\pi i}\int_{(1)} \mathbf{Z}_{-\mathcal{J},-\mathcal{I},t}(s)\frac{\mathbf{G}_{-\mathcal{J},-\mathcal{I},t}(s)}{\mathbf{G}_{\mathcal{I},\mathcal{J},t}(0)}\frac{G(s)}{s}ds \\ & + O((1+|t|)^{-A}).\end{aligned} \tag{2.14}$$

It follows from (2.12) and the definitions (2.6) and (2.7) that

$$\frac{\mathbf{G}_{\mathcal{I},\mathcal{J},t}(s)}{\mathbf{G}_{\mathcal{I},\mathcal{J},t}(0)} = \pi^{-3s} g_{\mathcal{I},\mathcal{J}}(s,t) \text{ and } \frac{\mathbf{G}_{-\mathcal{J},-\mathcal{I},t}(s)}{\mathbf{G}_{\mathcal{I},\mathcal{J},t}(0)} = \pi^{-3s} X_{\mathcal{I},\mathcal{J};t}\, g_{-\mathcal{J},-\mathcal{I}}(s,t). \tag{2.15}$$

The second equality makes use of the identity

$$X_{\mathcal{I},\mathcal{J};t} = \frac{\mathbf{G}_{-\mathcal{J},-\mathcal{I},t}(0)}{\mathbf{G}_{\mathcal{I},\mathcal{J},t}(0)}.$$

By definition $\mathbf{Z}_{\mathcal{I},\mathcal{J},t}(0) = \zeta_{\mathcal{I}}(\frac{1}{2}+it)\zeta_{\mathcal{J}}(\frac{1}{2}-it)$. Combining the above facts

$$\begin{aligned}\zeta_{\mathcal{I}}(\tfrac{1}{2}+it)\zeta_{\mathcal{J}}(\tfrac{1}{2}-it) = {} & \frac{1}{2\pi i}\int_{(1)} \mathbf{Z}_{\mathcal{I},\mathcal{J},t}(s)\pi^{-3s} g_{\mathcal{I},\mathcal{J}}(s,t)\frac{G(s)}{s}ds \\ & + \frac{1}{2\pi i}\int_{(1)} \mathbf{Z}_{-\mathcal{J},-\mathcal{I},t}(s)\pi^{-3s} X_{\mathcal{I},\mathcal{J};t}\, g_{-\mathcal{J},-\mathcal{I}}(s,t)\frac{G(s)}{s}ds + O((1+|t|)^{-A}).\end{aligned} \tag{2.16}$$

However, we have the Dirichlet series expansions

$$\mathbf{Z}_{\mathcal{I},\mathcal{J},t}(s) = \sum_{m,n=1}^{\infty} \frac{\sigma_{\mathcal{I}}(m)\sigma_{\mathcal{J}}(n)}{m^{\frac{1}{2}+s+it} n^{\frac{1}{2}+s-it}} \text{ and } \mathbf{Z}_{-\mathcal{J},-\mathcal{I},t}(s) = \sum_{m,n=1}^{\infty} \frac{\sigma_{-\mathcal{J}}(m)\sigma_{-\mathcal{I}}(n)}{m^{\frac{1}{2}+s+it} n^{\frac{1}{2}+s-it}}.$$

These expressions are inserted in (2.16). Since they are absolutely convergent on $\Re(s) = 1$, we may exchange integration and summation order. Thus by the definition (2.5) we arrive at (2.8). $\square$

The next lemma will give asymptotic estimates for the functions $X_{\mathcal{I},\mathcal{J};t}$ and $g_{\mathcal{I},\mathcal{J}}(s,t)$.

Lemma 2.2. *As $t \to \infty$*

(i) $X_{\mathcal{I},\mathcal{J};t} = \left(\frac{t}{2\pi}\right)^{-\sum_{i=1}^{3}(a_i+b_i)}(1+O(t^{-1}))$;

(ii) $g_{\mathcal{I},\mathcal{J}}(s,t) = \left(\frac{t}{2}\right)^{3s}(1+O(|s|^2 t^{-1}))$;

(iii) *For $x > t^3$, $V_{\mathcal{I},\mathcal{J}}(x) = O\Big(\Big(\dfrac{t^3}{x}\Big)^A\Big)$, where A is the constant given in Proposition* (2.1)*.*

Proof. The first two parts follow from Stirling's formula and are technical calculations. Since the proof of (i) is similar and easier than (ii), we leave it as an exercise. Their proof of (ii) will be deferred to Section 8 which contains Appendix 2. Proof of part (iii). Note that we can move the contour right to $\Re(s) = A$ so that

$$\begin{aligned} V_{\mathcal{I},\mathcal{J}}(x) &= \frac{1}{2\pi i}\int_{(A)} \frac{G(s)}{s} g(s,t) x^{-s} ds \\ &\ll \int_{A-i\infty}^{A+i\infty} \frac{|G(s)|}{|s|}\Big(\frac{t^3}{8x}\Big)^{\Re(s)}\Big(1+\frac{c|s|^2}{t}\Big)|ds| \end{aligned} \tag{2.17}$$

for some positive constant c, by part (ii). It follows that $V_{\mathcal{I},\mathcal{J}}(x) \ll (\frac{t^3}{x})^A$ as desired. ☐

In our evaluation of $I_{\mathcal{I},\mathcal{J};\omega}(T)$ we shall encounter the the Dirichlet series $\mathcal{Z}_{\mathcal{I},\mathcal{J}}(s) = \sum_{n=1}^{\infty} \frac{\sigma_{\mathcal{I}}(n)\sigma_{\mathcal{J}}(n)}{n^{1+s}}$. We now provide a factorization of this series into zeta factors times an absolutely convergent product near $s = 0$.

Lemma 2.3. *Let $\mathcal{I} = \{a_1, a_2, a_3\}$ and $\mathcal{J} = \{b_1, b_2, b_3\}$. We have that*

$$\mathcal{Z}_{\mathcal{I},\mathcal{J}}(s) = \Big(\prod_{i,j=1}^{3} \zeta(1+s+a_i+b_j)\Big)\mathcal{A}_{\mathcal{I},\mathcal{J}}(s)$$

where

$$\mathcal{A}_{\mathcal{I},\mathcal{J}}(s) = \prod_p \mathcal{A}_{p;\mathcal{I},\mathcal{J}}(s) \tag{2.18}$$

and

$$\mathcal{A}_{p;\mathcal{I},\mathcal{J}}(s) = P(p^{-a_1}, p^{-a_2}, p^{-a_3}, p^{-b_1}, p^{-b_2}, p^{-b_3}; p^{-s-1}) \tag{2.19}$$

where

$$\begin{aligned} &P(X_1,X_2,X_3,Y_1,Y_2,Y_3,U) \\ &= 1 - X_1X_2X_3Y_1Y_2Y_3(X_1^{-1}+X_2^{-1}+X_3^{-1})(Y_1^{-1}+Y_2^{-1}+Y_3^{-1})U^2 \\ &+ X_1X_2X_3Y_1Y_2Y_3 \times \\ &\Big((X_1^{-1}+X_2^{-1}+X_3^{-1})(X_1+X_2+X_3)+(Y_1^{-1}+Y_2^{-1}+Y_3^{-1})(Y_1+Y_2+Y_3)-2\Big)U^3 \\ &- X_1X_2X_3Y_1Y_2Y_3(X_1+X_2+X_3)(Y_1+Y_2+Y_3)U^4 \\ &+ (X_1X_2X_3Y_1Y_2Y_3)^2U^6. \end{aligned} \tag{2.20}$$

Observe that this implies that $\mathcal{A}_{\mathcal{I},\mathcal{J}}(s)$ is absolutely convergent in $\Re(s) > -\frac{1}{2}$ since for every $\varepsilon > 0$ $\mathcal{A}_{p;\mathcal{I},\mathcal{J}}(s) = 1 + O(p^{\varepsilon-2-2\sigma})$.

Proof. Let $s = 2z+1$. It is shown in [8] that

$$\sum_{n=1}^{\infty} \frac{\sigma_{\mathcal{I}}(n)\sigma_{\mathcal{J}}(n)}{n^{2z}} = \Big(\prod_{i,j=1}^{3}\zeta(2z+a_i+b_j)\Big)\mathcal{B}_{\mathcal{I},\mathcal{J}}(z)$$

where

$$\mathcal{B}_{\mathcal{I},\mathcal{J}}(z) = \prod_p \sum_{m=1}^{3} \prod_{i\neq m} \frac{\prod_{j=1}^{3}\Big(1-\frac{1}{p^{2z+a_i+b_j}}\Big)}{1-p^{b_m-b_i}} \tag{2.21}$$

It follows that $\mathcal{A}_{\mathcal{I},\mathcal{J}}(s) = \prod_p \mathcal{A}_{p;\mathcal{I},\mathcal{J}}(s)$ where

$$\mathcal{A}_{p;\mathcal{I},\mathcal{J}}(s) = \sum_{m=1}^{3} \prod_{i\neq m} \frac{\prod_{j=1}^{3}\left(1-\frac{1}{p^{1+s+a_i+b_j}}\right)}{1-p^{b_m-b_i}}. \tag{2.22}$$

It is proven in [8, p.66] that (2.22) is a polynomial in the p^{-a_i}, p^{-b_j}, and p^{-1-s}. Moreover, this polynomial is explicitly given in [8, eq.2.67, p.64] and is exactly (2.20). $\square$

3. A formula for the sixth moment

We now possess all the tools and lemmas to commence with our evaluation of $I_{\mathcal{I},\mathcal{J};\omega}(T)$. At the outset we assume that the elements of $\mathcal{I}$ and $\mathcal{J}$ are all distinct. By Lemma 2.1 it follows that

$$\begin{aligned} I_{\mathcal{I},\mathcal{J};\omega}(T) &= \sum_{m,n=1}^{\infty} \frac{\sigma_{\mathcal{I}}(m)\sigma_{\mathcal{J}}(n)}{(mn)^{\frac12}} \int_{-\infty}^{\infty} \left(\frac{m}{n}\right)^{-it} V_{\mathcal{I},\mathcal{J};t}(\pi^3 mn)\omega(t)dt \\ &+ \sum_{m,n=1}^{\infty} \frac{\sigma_{-\mathcal{J}}(m)\sigma_{-\mathcal{I}}(n)}{(mn)^{\frac12}} \int_{-\infty}^{\infty} \left(\frac{m}{n}\right)^{-it} X_{\mathcal{I},\mathcal{J};t} V_{-\mathcal{J},-\mathcal{I};t}(\pi^3 mn)\omega(t)dt + O\Big(\int_{-\infty}^{\infty}\omega(t)t^{-1}dt\Big) \\ &:= I^{(1)} + I^{(2)} + O(1). \end{aligned}$$

Opening the integral formula for V yields

$$I^{(1)} = \sum_{m,n=1}^{\infty} \frac{\sigma_{\mathcal{I}}(m)\sigma_{\mathcal{J}}(n)}{(mn)^{\frac12}} \frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s}(\pi^3 mn)^{-s} \int_{-\infty}^{\infty} \left(\frac{m}{n}\right)^{-it} g_{\mathcal{I},\mathcal{J}}(s,t)\omega(t)dtds, \tag{3.1}$$

and

$$I^{(2)} = \sum_{m,n=1}^{\infty} \frac{\sigma_{-\mathcal{J}}(m)\sigma_{-\mathcal{I}}(n)}{(mn)^{\frac12}} \frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s}(\pi^3 mn)^{-s} \int_{-\infty}^{\infty} \left(\frac{m}{n}\right)^{-it} X_{\mathcal{I},\mathcal{J};t}\; g_{-\mathcal{J},-\mathcal{I}}(s,t)\omega(t)dtds. \tag{3.2}$$

We now define the diagonal terms $I_D^{(1)}$ and $I_D^{(2)}$ to be those terms above where $m = n$. Likewise the off-diagonal terms $I_O^{(1)}$ and $I_O^{(2)}$ are those terms above where $m \neq n$. More precisely,

$$I_D^{(1)} = \sum_{n=1}^{\infty} \frac{\sigma_{\mathcal{I}}(n)\sigma_{\mathcal{J}}(n)}{n} \frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s}(\pi^3 n^2)^{-s} \int_{-\infty}^{\infty} g_{\mathcal{I},\mathcal{J}}(s,t)\omega(t)dtds, \tag{3.3}$$

$$I_D^{(2)} = \sum_{n=1}^{\infty} \frac{\sigma_{-\mathcal{J}}(n)\sigma_{-\mathcal{I}}(n)}{n} \frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s}(\pi^3 n^2)^{-s} \int_{-\infty}^{\infty} X_{\mathcal{I},\mathcal{J};t}\; g_{-\mathcal{J},-\mathcal{I}}(s,t)\omega(t)dtds, \tag{3.4}$$

$$I_O^{(1)} = \sum_{m\neq n} \frac{\sigma_{\mathcal{I}}(m)\sigma_{\mathcal{J}}(n)}{(mn)^{\frac12}} \frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s}(\pi^3 mn)^{-s} \int_{-\infty}^{\infty} \left(\frac{m}{n}\right)^{-it} g_{\mathcal{I},\mathcal{J}}(s,t)\omega(t)dtds, \tag{3.5}$$

$$I_O^{(2)} = \sum_{m\neq n} \frac{\sigma_{-\mathcal{J}}(m)\sigma_{-\mathcal{I}}(n)}{(mn)^{\frac12}} \frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s}(\pi^3 mn)^{-s} \int_{-\infty}^{\infty} \left(\frac{m}{n}\right)^{-it} X_{\mathcal{I},\mathcal{J};t}\; g_{-\mathcal{J},-\mathcal{I}}(s,t)\omega(t)dtds. \tag{3.6}$$

Summarizing, we have

$$I^{(j)} = I_D^{(j)} + I_O^{(j)} \text{ for } j = 1,2 \tag{3.7}$$

and thus

$$I_{\mathcal{I},\mathcal{J};\omega}(T) = (I_D^{(1)} + I_O^{(1)}) + (I_D^{(2)} + I_O^{(2)}) + O(1). \tag{3.8}$$

The asymptotic evaluation of $I_{\mathcal{I},\mathcal{J};\omega}(T)$ is reduced to evaluating $I_D^{(j)}$ and $I_O^{(j)}$. The calculations of the $I_D^{(j)}$ are straightforward. The majority of this article concerns the evaluation of the off-diagonal sums $I_O^{(j)}$.

4. Diagonal terms

In this section we evaluate the diagonal terms $I_D^{(j)}$.

Proposition 4.1. *For every $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that for $T \geq T_\varepsilon$*

$$\begin{aligned} I_D^{(1)} &= \int_{-\infty}^{\infty} \mathcal{Z}_{\mathcal{I},\mathcal{J}}(0)\omega(t)dt \\ &\quad + \sum_{i,j=1}^{3} \operatorname{Res}_{s=\frac{-a_i-b_j}{2}} \mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s)\frac{G\left(\frac{-a_i-b_j}{2}\right)}{\frac{-a_i-b_j}{2}} \int_{-\infty}^{\infty} \omega(t)\left(\frac{t}{2\pi}\right)^{-\frac{3}{2}(a_i+b_j)} dt + O(T^{\frac{1}{4}+\varepsilon}) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} I_D^{(2)} &= \int_{-\infty}^{\infty} \left(\frac{t}{2\pi}\right)^{-\sum_{k=1}^{3}(a_k+b_k)} \mathcal{Z}_{-\mathcal{J},-\mathcal{I}}(0)\omega(t)dt \\ &\quad + \sum_{i,j=1}^{3} \operatorname{Res}_{s=\frac{a_i+b_j}{2}} \mathcal{Z}_{-\mathcal{J},-\mathcal{I}}(2s)\frac{G\left(\frac{a_i+b_j}{2}\right)}{\frac{a_i+b_j}{2}} \int_{-\infty}^{\infty} \omega(t)\left(\frac{t}{2\pi}\right)^{-\sum_{k=1}^{3}(a_k+b_k)+\frac{3}{2}(a_i+b_j)} dt + O(T^{\frac{1}{4}+\varepsilon}). \end{aligned} \tag{4.2}$$

Proof. By (3.3), moving the sum inside the integral,

$$\begin{aligned} I_D^{(1)} &= \int_{-\infty}^{\infty} \omega(t)\frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s}\pi^{-3s} g_{\mathcal{I},\mathcal{J}}(s,t)\sum_{n=1}^{\infty}\frac{\sigma_{\mathcal{I}}(n)\sigma_{\mathcal{J}}(n)}{n^{1+2s}}dsdt \\ &= \int_{-\infty}^{\infty} \omega(t)\frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s}\pi^{-3s} g_{\mathcal{I},\mathcal{J}}(s,t)\mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s)dsdt. \end{aligned}$$

By Lemma 2.2 (ii)

$$I_D^{(1)} = \int_{-\infty}^{\infty} \omega(t)\frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s}\left(\frac{t}{2\pi}\right)^{3s}(1+O(|s|^2|t|^{-1}))\mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s)dsdt. \tag{4.3}$$

By Cauchy's theorem, we move the s integral to the $\Re(s) = \varepsilon$ line where $\varepsilon > 0$. On this line the contribution from the $O(|s|^2|t|^{-1})$ term is

$$\ll_\varepsilon |t|^{3\varepsilon-1}\int_{-\infty}^{\infty} \frac{|G(\varepsilon+iu)|}{|\varepsilon+iu|}|\varepsilon+iu|^2du \ll |t|^{3\varepsilon-1},$$

since G is of rapid decay. Since $\int_{-\infty}^{\infty}\omega(t)|t|^{3\varepsilon-1}dt \ll T^{3\varepsilon}$, it follows that

$$I_D^{(1)} = \int_{-\infty}^{\infty} \omega(t)\frac{1}{2\pi i}\int_{(\varepsilon)} \frac{G(s)}{s}\left(\frac{t}{2\pi}\right)^{3s}\mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s)dsdt + O(T^{3\varepsilon}).$$

By Lemma $\mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s)$ has poles at

$$s = 0 \text{ and } s = \frac{-a_i-b_j}{2} \text{ for } i \in \{1,2,3\} \text{ and } j \in \{1,2,3\}. \tag{4.4}$$

We now move the line of integration to $\Re(s) = -\frac{1}{4}+\varepsilon$, crossing the poles listed in (4.4). The residue at $s = 0$ is

$$\int_{-\infty}^{\infty} \mathcal{Z}_{\mathcal{I},\mathcal{J}}(0)\omega(t)dt. \tag{4.5}$$

The residues at $s = \frac{-a_i-b_j}{2}$ are

$$\operatorname{Res}_{s=\frac{-a_i-b_j}{2}} \mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s) \frac{G\left(\frac{-a_i-b_j}{2}\right)}{\frac{-a_i-b_j}{2}} \int_{-\infty}^{\infty} \omega(t)\left(\frac{t}{2\pi}\right)^{-\frac{3}{2}(a_i+b_j)} dt. \tag{4.6}$$

The s-integral on the line $\Re(s) = -\frac{1}{4}+\varepsilon$ is

$$\frac{1}{2\pi i}\int_{(-\frac{1}{4}+\varepsilon)} \frac{G(s)}{s}\left(\frac{t}{2\pi}\right)^{3s} \mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s)ds \ll t^{-\frac{3}{4}+3\varepsilon}\int_{-\infty}^{\infty} \frac{|G(-\frac{1}{4}+\varepsilon+iu)|}{|-\frac{1}{4}+\varepsilon+iu|}|u|^B du \ll t^{-\frac{3}{4}+3\varepsilon}.$$

and thus

$$\int_{-\infty}^{\infty}\omega(t)\frac{1}{2\pi i}\int_{(-\frac{1}{4}+\varepsilon)} \frac{G(s)}{s}\left(\frac{t}{2\pi}\right)^{3s}\mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s)dsdt \ll \int_{T/2}^{4T} t^{-\frac{3}{4}+3\varepsilon}|\omega(t)|dt \ll T^{\frac{1}{4}+3\varepsilon}.$$

Combining (4.5), (4.6), and this last estimate completes the evaluation of $I_D^{(1)}$. The evaluation of $I_D^{(2)}$ can be done in a completely analogous fashion. For instance, we can show that

$$I_D^{(2)} = \int_{-\infty}^{\infty}\omega(t)\left(\frac{t}{2\pi}\right)^{-\sum_{k=1}^{3}(a_k+b_k)}\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}\left(\frac{t}{2\pi}\right)^{3s}\mathcal{Z}_{-\mathcal{J},-\mathcal{I}}(2s)dsdt + O(T^{3\varepsilon}).$$

This formula is obtained from (4.3) by formally replacing $\mathcal{I}$ by $-\mathcal{J}$ and by replacing $\mathcal{J}$ by $-\mathcal{I}$ and by inserting the factor $(\frac{t}{2\pi})^{-\sum_{k=1}^{3}(a_k+b_k)}$. Doing this we obtain

$$I_D^{(2)} = \int_{-\infty}^{\infty}\omega(t)\left(\frac{t}{2\pi}\right)^{-\sum_{k=1}^{3}(a_k+b_k)}\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}\left(\frac{t}{2\pi}\right)^{3s}\mathcal{Z}_{-\mathcal{J},-\mathcal{I}}(2s)dsdt + O(T^{3\varepsilon}).$$

As before we shall move the contour in the s-integral left, passing poles at

$$s = 0 \text{ and } s = \frac{a_i+b_j}{2} \text{ for } i,j = 1,\ldots,3.$$

Calculating the residues as before, we arrive at (4.2). □

5. Proof of main theorem and initial evaluation of the off-diagonal terms

In this section we begin the evaluation of the off-diagonal terms $I_O^{(j)}$. In addition, we shall prove Theorem 1.1. This is the most involved part of the argument. We aim to prove

Proposition 5.1. *For every $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that for $T \geq T_\varepsilon$*

$$\begin{aligned} I_O^{(1)} + I_O^{(2)} &= \int_{-\infty}^{\infty}\Big(\sum_{j=1}^{2}\sum_{\substack{\mathcal{S}\in\Phi_j\\ \mathcal{T}\in\Psi_j}}\mathcal{Z}_{\mathcal{I}_\mathcal{S},\mathcal{J}_\mathcal{T}}(0)\left(\frac{t}{2\pi}\right)^{-\mathcal{S}-\mathcal{T}}\Big)\omega(t)dt \\ &- \sum_{i,j=1}^{3}\operatorname{Res}_{s=\frac{-a_i-b_j}{2}}\mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s)\frac{G\left(\frac{-a_i-b_j}{2}\right)}{\frac{-a_i-b_j}{2}}\int_{-\infty}^{\infty}\omega(t)\left(\frac{t}{2\pi}\right)^{-\frac{3}{2}(a_i+b_j)}dt \\ &- \sum_{i,j=1}^{3}\operatorname{Res}_{s=\frac{a_i+b_j}{2}}\mathcal{Z}_{-\mathcal{J},-\mathcal{I}}(2s)\frac{G\left(\frac{a_i+b_j}{2}\right)}{\frac{a_i+b_j}{2}}\int_{-\infty}^{\infty}\omega(t)\left(\frac{t}{2\pi}\right)^{-\sum_{k=1}^{3}(a_k+b_k)+\frac{3}{2}(a_i+b_j)}dt \\ &+ O\Big(T^{\frac{3\vartheta}{2}+\varepsilon}\left(\frac{T}{T_0}\right)^{1+C} + \eta_C T^{\frac{3}{2}C-\frac{1}{2}+\frac{3\vartheta}{2}+\varepsilon}T_0^{-C} + T^{\frac{1}{2}+\varepsilon}\Big) \end{aligned} \tag{5.1}$$

uniformly for $|a_i|, |b_j| \ll (\log T)^{-1}$.

The main theorem, Theorem 1.1, follows from Proposition 4.1 and Proposition 5.1.

Proof of Theorem 1.1. Combining (4.1), (4.2), and (5.1)

$$I_{\mathcal{I},\mathcal{J};\omega}(T) = \int_{-\infty}^{\infty}\Big(\sum_{j=0}^{3}\sum_{\substack{\mathcal{S}\in\Phi_j\\ \mathcal{T}\in\Psi_j}} \mathcal{Z}_{\mathcal{I}_\mathcal{S},\mathcal{J}_\mathcal{T}}(0)\Big(\frac{t}{2\pi}\Big)^{-\mathcal{S}-\mathcal{T}}\Big)\omega(t)dt$$
$$+ O\Big(T^{\frac{3\vartheta}{2}+\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C} + \eta_C T^{\frac{3}{2}C-\frac{1}{2}+\frac{3\vartheta}{2}+\varepsilon}T_0^{-C} + T^{\frac{1}{2}+\varepsilon}\Big). \tag{5.2}$$

Notice that the sum of residues in (4.1) and (4.2) exactly cancel the two sums of residues in (5.1). Also, the first terms in (4.1) and (4.2) are added into the first sum of (5.1) making the sum over $j \in \{0,1,2,3\}$. This result is valid if $\mathcal{I}$ and $\mathcal{J}$ consist of distinct elements. However, $I_{\mathcal{I},\mathcal{J};\omega}(T)$ is holomorphic if the a_i's and b_j's satisfy $|a_i| < \frac{1}{2}$ and $|b_j| < \frac{1}{2}$. In addition, by Lemma 2.5.1 of [8] the first term after the equality in (5.2) is holomorphic if $|a_i| < \delta$ and $|b_j| < \delta$ for a sufficiently small δ. It follows that the error term in (5.2) is holomorphic in the a_i, b_j, as long they are restricted to small enough disks and is thus continuous in the a_i and b_j. By a continuity argument it follows that (5.2) holds in the case that $\mathcal{I}$ and $\mathcal{J}$ do not consist of distinct elements. This completes the proof of the main theorem. □

We now begin an initial evaluation of $I_O^{(1)}$. The evaluation of $I_O^{(2)}$ will be similar and we shall only mention the minor differences in the argument. Let

$$f^*(x,y) = \frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3 xy}\Big)^s\frac{1}{T}\int_{-\infty}^{\infty}\Big(\frac{x}{y}\Big)^{-it}g(s,t)\omega(t)dt, \tag{5.3}$$

so that

$$I_O^{(1)} = T\sum_{m\neq n}\frac{\sigma_I(m)\sigma_\mathcal{J}(n)}{\sqrt{mn}}f^*(m,n).$$

We now introduce a smooth partition of unity to simplify the evaluation of this sum. The main reason this is done is to apply a version of the additive divisor conjecture where the variables are restricted to boxes of the shape $[M,2M]\times[N,2N]$ with $M \asymp N$. Let W_0 be a smooth function supported in $[1,2]$ such that

$$\sum_M W_0\Big(\frac{x}{M}\Big) = 1 \tag{5.4}$$

where M runs through a sequence of real numbers such that $\#\{M \mid M \le X\} \ll \log X$. See [21] for an example of such a function. Thus

$$I_O^{(1)} = \sum_{M,N} I_{M,N} \tag{5.5}$$

where

$$I_{M,N} = \frac{T}{\sqrt{MN}}\sum_{m\neq n}\sigma_\mathcal{I}(m)\sigma_\mathcal{J}(n)W\Big(\frac{m}{M}\Big)W\Big(\frac{n}{N}\Big)f^*(m,n) \tag{5.6}$$

and $W(x) = x^{-\frac{1}{2}}W_0(x)$. Note that we may assume $MN \le T^{3+\varepsilon}$ by Lemma 2.2 (iii). Observe that $f^*(x,y)$ is small unless x and y close to each other, due to the cancellation in $(x/y)^{-it}$. This is since

$$\frac{1}{T}\int_{-\infty}^{\infty}\Big(\frac{x}{y}\Big)^{-it}g(s,t)\omega(t)dt \ll \frac{1}{T|\log(x/y)|^j}\int_{-\infty}^{\infty}\Big|\frac{\partial^j}{\partial t^j}g(s,t)\omega(t)\Big| \ll \frac{P_j(|s|)T^{3\Re(s)}}{|\log(x/y)|^jT_0^j},$$

where we have integrated by parts j times. Therefore

$$f^*(x,y) \ll \frac{T^{3\varepsilon}}{(xy)^\varepsilon|\log(x/y)|^jT_0^j}\int_{(\varepsilon)}\frac{|G(s)|}{|s|}P_j(|s|)|ds| \ll \frac{T^{3\varepsilon}}{|\log(x/y)|^jT_0^j}.$$

If

$$|\log(x/y)| \gg T_0^{-1+\varepsilon}, \tag{5.7}$$

then for any $A > 0$ we obtain

$$f^*(x, y) \ll T^{3\varepsilon-\varepsilon j} \ll T^{-A}$$

by choosing $j > (A + 3)/\varepsilon$. Letting $m - n = r$, it follows that

$$I_{M,N} = \frac{T}{\sqrt{MN}} \sum_{r\neq 0} \sum_{\substack{m-n=r \\ |\log(\frac{m}{n})| \ll T_0^{-1+\varepsilon}}} \sigma_{\mathfrak{I}}(m)\sigma_{\mathfrak{J}}(n) W\Big(\frac{m}{M}\Big) W\Big(\frac{n}{N}\Big) f^*(m, n) + O(T^{-A}).$$

Note that the condition (5.7) implies that $\frac{N}{3} \leq M \leq 3N$. Note that if $M < N/3$ or $M > 3N$, then $|\log \frac{m}{n}| \geq \log(3/2)$. Also observe that (5.7) implies that $M, N \gg T_0^{1-\varepsilon}$. For the rest of the article we shall write $M \asymp N$ to mean that $\frac{N}{3} \leq M \leq 3N$. Thus we shall be restricted to evaluating $I_{M,N}$ in the case where $M \asymp N$. If $x - y = r$ then

$$f^*(x, y) = \frac{1}{2\pi i} \int_{(\varepsilon)} \frac{G(s)}{s} \Big(\frac{1}{\pi^3 xy}\Big)^s \frac{1}{T} \int_{-\infty}^{\infty} \Big(1 + \frac{r}{y}\Big)^{-it} g(s, t)\omega(t) dt ds. \tag{5.8}$$

We have thus shown

Proposition 5.2. *Let $A > 0$ be arbitrary and fixed. Then we have for $M \asymp N$*

$$I_{M,N} = \frac{T}{\sqrt{MN}} \sum_{0<|r|\ll \frac{\sqrt{MN}}{T_0} T^{\varepsilon}} \sum_{m-n=r} \sigma_{\mathfrak{I}}(m)\sigma_{\mathfrak{J}}(n) f(m, n) + O(T^{-A})$$

where

$$f(x, y) = W\Big(\frac{x}{M}\Big) W\Big(\frac{y}{N}\Big) \frac{1}{2\pi i} \int_{(\varepsilon)} \frac{G(s)}{s} \Big(\frac{1}{\pi^3 xy}\Big)^s \frac{1}{T} \int_{-\infty}^{\infty} \Big(1 + \frac{r}{y}\Big)^{-it} g(s, t)\omega(t) dt\, ds. \tag{5.9}$$

For $M \not\asymp N$, we have $I_{M,N} \ll T^{-A}$.

In summary, we have established

$$I_O^{(1)} = \sum_{\substack{M,N \\ M\asymp N, MN \ll T^{3+\varepsilon} \\ M,N \gg T_0^{1-\varepsilon}}} I_{M,N}. \tag{5.10}$$

Note that the condition $MN \ll T^{3+\varepsilon}$ can be added in by Lemma 2.2 (iii) which shows that $V_{\mathfrak{I},\mathfrak{J}}$ is very small if $MN \gg T^{3+\varepsilon}$. We are now in a position to estimate $I_{M,N}$ by the conjecture for the ternary additive divisor problem. It suffices to verify that f satisfies the following condition on its partial derivatives.

Lemma 5.3. *We have for $M \asymp N$ that*

$$x^i y^j f^{(i,j)}(x, y) \ll P^{i+j} \tag{5.11}$$

where $P := (\frac{M}{rT_0} + \frac{T}{T_0})T^{\varepsilon}$.

This proof of this technical lemma is deferred to Section 8 (Appendix 2). By Conjecture 2

$$I_{M,N} = \frac{T}{\sqrt{MN}} \sum_{0<|r|\ll \frac{\sqrt{MN}}{T_0}T^\varepsilon} \sum_{i_1=1}^{3}\sum_{i_2=1}^{3} \prod_{j_1\neq i_1} \zeta(1-a_{i_1}+a_{j_1}) \prod_{j_2\neq i_2} \zeta(1-b_{i_2}+b_{j_2})$$
$$\times \sum_{\ell=1}^{\infty} \frac{c_\ell(r) G_{\mathcal{J}}(1-a_{i_1},\ell) G_{\mathcal{J}}(1-b_{i_2},\ell)}{\ell^{2-a_{i_1}-b_{i_2}}} \int_{\max(0,r)}^{\infty} f(x,x-r) x^{-a_{i_1}} (x-r)^{-b_{i_2}}\, dx + O(\mathcal{E}_{M,N})$$

where

$$\mathcal{E}_{M,N} = \frac{T}{\sqrt{MN}} \sum_{0<|r|\ll \frac{\sqrt{MN}}{T_0}T^\varepsilon} \Big(\frac{M}{rT_0} + \frac{T}{T_0}\Big)^C M^{\vartheta+\varepsilon}. \tag{5.12}$$

Next we consider the contribution of the errors $\mathcal{E}_{M,N}$ to (5.5).

Lemma 5.4. *Let $\varepsilon > 0$ and T is sufficiently large with respect to ε. Then*

$$\sum_{\substack{M,N \\ M\asymp N, MN\ll T^{3+\varepsilon}}} \mathcal{E}_{M,N} \ll T^{\frac{3\vartheta}{2}+\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C} + \eta_C T^{\frac{3}{2}C-\frac{1}{2}+\frac{3\vartheta}{2}+\varepsilon} T_0^{-C}. \tag{5.13}$$

where $\eta_C = 1$ if $C \geq 1$ and $\eta_C = 0$ if $C < 1$.

Proof. We now estimate sum in the error term. Since $M \asymp N$, we have

$$\begin{aligned}\mathcal{E}_{M,N} &\ll \frac{T}{M} \sum_{0<|r|\ll \frac{M}{T_0}T^\varepsilon} T^\varepsilon \Big(\frac{M}{rT_0} + \frac{T}{T_0}\Big)^C M^{\vartheta+\varepsilon} \\ &\ll T^\varepsilon \frac{T}{M}\Big(\sum_{0<|r|\ll \frac{M}{T_0}T^\varepsilon} \Big(\frac{M}{rT_0}\Big)^C M^{\vartheta+\varepsilon} + \sum_{0<|r|\ll \frac{M}{T_0}T^\varepsilon} \Big(\frac{T}{T_0}\Big)^C M^{\vartheta+\varepsilon}\Big),\end{aligned}$$

as $(x+y)^C \ll_C x^C + y^C$. Thus

$$\begin{aligned}\mathcal{E}_{M,N} &\ll T^\varepsilon \frac{T}{M}\Big(\Big(\frac{M}{T_0}\Big)^C M^{\vartheta+\varepsilon} \sum_{0<|r|\ll \frac{M}{T_0}T^\varepsilon} r^{-C} + \Big(\frac{T}{T_0}\Big)^C M^{\vartheta+\varepsilon} \frac{M}{T_0} T^\varepsilon\Big) \\ &\ll T^\varepsilon \frac{T}{M}\Big(\Big(\frac{M}{T_0}\Big)^C M^{\vartheta+\varepsilon} \sum_{0<|r|\ll \frac{M}{T_0}T^\varepsilon} r^{-C}\Big) + T^\varepsilon \Big(\frac{T}{T_0}\Big)^{1+C} M^{\vartheta+\varepsilon}.\end{aligned}$$

Observe that

$$\sum_{0<|r|\ll \frac{M}{T_0}T^\varepsilon} r^{-C} \ll \begin{cases} (\frac{M}{T_0}T^\varepsilon)^{1-C} & \text{if } 0 < C < 1, \\ \log T & \text{if } C \geq 1. \end{cases} \tag{5.14}$$

Therefore, if $0 < C < 1$, then

$$\begin{aligned}\mathcal{E}_{M,N} &\ll T^\varepsilon \frac{T}{M}\Big(\frac{M}{T_0}\Big)^C M^{\vartheta+\varepsilon}\Big(\frac{M}{T_0}T^\varepsilon\Big)^{1-C} + T^\varepsilon\Big(\frac{T}{T_0}\Big)^{1+C} M^{\vartheta+\varepsilon} \\ &= T^{\varepsilon(2-C)} \frac{T}{T_0} M^{\vartheta+\varepsilon} + T^\varepsilon \Big(\frac{T}{T_0}\Big)^{1+C} M^{\vartheta+\varepsilon} \ll T^\varepsilon\Big(\frac{T}{T_0}\Big)^{1+C} M^{\vartheta+\varepsilon}\end{aligned} \tag{5.15}$$

and it $C \geq 1$,

$$\mathcal{E}_{M,N} \ll T^\varepsilon \frac{T}{M}\Big(\frac{M}{T_0}\Big)^C M^{\vartheta+\varepsilon} + T^\varepsilon\Big(\frac{T}{T_0}\Big)^{1+C} M^{\vartheta+\varepsilon}. \tag{5.16}$$

We have the bounds

$$\sum_{\substack{MN\ll T^{3+\varepsilon}\\ M\asymp N}} T^{\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C} M^{\vartheta+\varepsilon} \ll T^{\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C} \sum_{M\ll T^{\frac{3}{2}+\varepsilon}} \sum_{N\asymp M} M^{\vartheta+\varepsilon}$$
$$\ll T^{\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C} (T^{\frac{3}{2}+\varepsilon})^{\vartheta+\varepsilon}\log^2 T \ll T^{\frac{3\vartheta}{2}+\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C} \tag{5.17}$$

and

(5.18)
$$\sum_{\substack{MN\ll T^{3+\varepsilon}\\ M\asymp N}} T^{\varepsilon}\frac{T}{M}\Big(\frac{M}{T_0}\Big)^{C} M^{\vartheta+\varepsilon} \ll \frac{T^{1+\varepsilon}}{T_0^C}\sum_{M\ll T^{\frac{3}{2}+\varepsilon}}\sum_{N\asymp M} M^{C-1+\vartheta+\varepsilon} \ll \frac{T^{1+\varepsilon}}{T_0^C}(T^{\frac{3}{2}+\varepsilon})^{C-1+\vartheta}T^{\varepsilon} = T^{\frac{3}{2}C-\frac{1}{2}+\frac{3\vartheta}{2}+\varepsilon}T_0^{-C}.$$

Therefore (5.13) follows from combining (5.15), (5.16), (5.17), and (5.18). $\square$

This leads to the following result.

Proposition 5.5.

$$I_O^{(1)} = \sum_{i_1=1}^{3}\sum_{i_2=1}^{3} I_{(i_1,i_2)}^{(1)} + O\Big(T^{\frac{3\vartheta}{2}+\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C} + \eta_C T^{\frac{3}{2}C-\frac{1}{2}+\frac{3\vartheta}{2}+\varepsilon}T_0^{-C}\Big) \tag{5.19}$$

where

$$I_{(i_1,i_2)}^{(1)} = \sum_{\substack{M,N\\ M\asymp N}}\sum_{r\neq 0}\frac{T}{\sqrt{MN}}\prod_{j_1\neq i_1}\zeta(1-a_{i_1}+a_{j_1})\prod_{j_2\neq i_2}\zeta(1-b_{i_2}+b_{j_2})\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathcal{I}}(1-a_{i_1},l)G_{\mathcal{J}}(1-b_{i_2},l)}{l^{2-a_{i_1}-b_{i_2}}}$$
$$\times\int_{\max(0,r)}^{\infty} f(x,x-r)x^{-a_{i_1}}(x-r)^{-b_{i_2}}\,dx.$$

Note that we can add back in those M and N not satisfying $M\asymp N$ by the decay of $f^*(x,y)$.

6. Further evaluation of $I_O^{(1)}$. Evaluation of $I_{(i_1,i_2)}^{(1)}$

By (6.6) the evaluation of $I_O^{(1)}$ has been reduced to the evaluation of $I_{(i_1,i_2)}^{(1)}$. In this calculation we shall encounter the Dirichlet series

$$H_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(s) = \sum_{r=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathcal{I}}(1-a_1,\ell)G_{\mathcal{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}r^{a_1+b_1+2s}}. \tag{6.1}$$

Moreover, $H_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(s)$ equals a product of ζ functions times a nice infinite product $\mathcal{C}_{\mathcal{I},\mathcal{J};a_1,b_1}(s)$.

Proposition 6.1. *(i) For $\Re(s)>\frac{1}{2}$,*

$$H_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(s) = \zeta(a_1+b_1+2s)\prod_{\substack{k_1\neq 1\\ k_2\neq 1}}\zeta(1+a_{k_1}+b_{k_2}+2s)\mathcal{C}_{\mathcal{I},\mathcal{J};a_1,b_1}(s)$$

where

$$\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(s) = \prod_p \mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(p;s), \tag{6.2}$$

$$\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(p;s) = Q(p^{-a_2},p^{-a_3},p^{-b_2},p^{-b_3};p^{-a_1},p^{-b_1};p^{-1},p^{-2s}), \tag{6.3}$$

and

(6.4)

$$Q(X_2,X_3,Y_2,Y_3;X_1,Y_1;U,V)=\Bigg(1+\Bigg(\frac{UVX_2Y_2}{1-UVX_2Y_2}\frac{1-UX_3X_1^{-1}}{1-X_3X_2^{-1}}\frac{1-UY_3Y_1^{-1}}{1-Y_3Y_2^{-1}}$$
$$+\frac{UVX_2Y_3}{1-UVX_2Y_3}\frac{1-UX_3X_1^{-1}}{1-X_3X_2^{-1}}\frac{1-UY_2Y_1^{-1}}{1-Y_2Y_3^{-1}}$$
$$+\frac{UVX_3Y_2}{1-UVX_3Y_2}\frac{1-UX_2X_1^{-1}}{1-X_2X_3^{-1}}\frac{1-UY_3Y_1^{-1}}{1-Y_3Y_2^{-1}}$$
$$+\frac{UVX_3Y_3}{1-UVX_3Y_3}\frac{1-UX_2X_1^{-1}}{1-X_2X_3^{-1}}\frac{1-UY_2Y_1^{-1}}{1-Y_2Y_3^{-1}}\Bigg)(1-UV^{-1}X_1^{-1}Y_1^{-1})\Bigg)$$
$$\times(1-UVX_2Y_2)(1-UVX_2Y_3)(1-UVX_3Y_2)(1-UVX_3Y_3).$$

Moreover, we have

$$\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(p;s)=1+O(p^{-2-2\sigma}) \tag{6.5}$$

and hence $\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(s)$ *is absolutely convergent for* $\Re(s)>-\frac{1}{2}$.

The following proposition shows that the local factors $\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(p;s)$ satisfy certain identities at special values of s, relating them to the local factors $A_{p;\mathcal{I},\mathcal{J}}(s)$ which occur in Lemma 4.

Proposition 6.2. *Let* $\mathcal{I}=\{a_1,a_2,a_3\}$ *and* $\mathcal{J}=\{b_1,b_2,b_3\}$. *We have the identitites*

(i) $\mathcal{A}_{\mathcal{I}_{\{a_1\}},\mathcal{J}_{\{b_1\}}}(0)=\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(0)$,
(ii) $\mathcal{A}_{\mathcal{I},\mathcal{J}}(-a_1-b_1)=\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\frac{a_1+b_1}{2})$,
(iii) $\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\frac{a_2+b_2}{2})=\mathcal{C}_{-\mathcal{J},-\mathcal{I};\{-b_3\},\{-a_3\}}(\frac{b_2+a_2}{2})$.

This proposition shall be demonstrated in Section 7 (Appendix 1). These identities shall be reduced to polynomial identities. Alternately parts (i) and (ii) also follow from an identity in [15, Sections 3,4]. In fact, their argument establishes such identities for general sets $\mathcal{I}=\{a_1,a_2,\ldots,a_k\}$ and $\mathcal{J}=\{b_1,b_2,\ldots,b_k\}$. Based on the previous propositions we shall establish the following formulae.

Proposition 6.3. *We have*

$$I_O^{(1)}=\sum_{i_1=1}^{3}\sum_{i_2=1}^{3}I^{(1)}_{(i_1,i_2)}+O\Big(T^{\frac{3\vartheta}{2}+\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C}+\eta_C T^{\frac{3}{2}C-\frac{1}{2}+\frac{3\vartheta}{2}+\varepsilon}T_0^{-C}\Big). \tag{6.6}$$

where

$$\begin{aligned}I^{(1)}_{(i_1,i_2)}=&\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-a_{i_1}-b_{i_2}}\mathcal{Z}_{\mathcal{I}_{\{a_{i_1}\}},\mathcal{J}_{\{b_{i_2}\}}}(0)dt\\ &-\frac{1}{2}\mathrm{Res}_{s=\frac{-a_{i_1}-b_{i_2}}{2}}\mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s)\frac{G\Big(\frac{-a_{i_1}-b_{i_2}}{2}\Big)}{\frac{-a_{i_1}-b_{i_2}}{2}}\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-\frac{3}{2}(a_{i_1}+b_{i_2})}dt\\ &+\sum_{\substack{(k_1,k_2)\\ k_1\neq i_1,k_2\neq i_2}}\mathcal{T}_{(i_1,i_2);(k_1,k_2)}+O(T^{\frac{1}{2}+\varepsilon})\end{aligned} \tag{6.7}$$

and

$$\mathcal{T}_{(i_1,i_2);(k_1,k_2)} = \frac{1}{2}\prod_{j_1\neq i_1}\zeta(1-a_{i_1}+a_{j_1})\prod_{j_2\neq i_2}\zeta(1-b_{i_2}+b_{j_2})\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-a_{i_1}-b_{i_2}-\frac{a_{k_1}+b_{k_2}}{2}}dt$$
$$\times\Big(\prod_{\substack{l_1\neq i_1, l_2\neq i_2\\ (l_1,l_2)\neq(k_1,k_2)}}\zeta(1+a_{l_1}+b_{l_2}-a_{k_1}-b_{k_2})\Big)\zeta(1-a_{i_1}-b_{i_2}+a_{k_2}+b_{k_2})$$
$$\times\,\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_{i_1}\},\{b_{i_2}\}}\big(-\tfrac{a_{k_1}+b_{k_2}}{2}\big)\frac{G(-\frac{a_{k_1}+b_{k_2}}{2})}{-\frac{a_{k_1}+b_{k_2}}{2}} \tag{6.8}$$

Remark. It should be observed that the formulae for $I^{(1)}_{(i_1,i_2)}$ contains the extra unwanted residues $\mathcal{T}_{(i_1,i_2);(k_1,k_2)}$. Notice that these do not appear in the formula for $I^{(1)}_O + I^{(2)}_O$ given by (5.1).

We can also prove an analogous result for $I^{(2)}_O$. Note that by (3.6) and (3.5) we see that $I^{(2)}_O$ is the same as $I^{(1)}_O$, except that $\mathcal{I}\to-\mathcal{J}$ and $\mathcal{J}\to-\mathcal{I}$ and there is the additional factor of $X_{\mathcal{I},\mathcal{J};t}$. It follows that we may obtain (6.9) from (6.9) by replacing each $\mathcal{I}$ by $\mathcal{J}$, each $\mathcal{J}$ by $\mathcal{I}$, and inserting a factor $(\frac{t}{2\pi})^{-\sum_{i=1}^{3}(a_i+b_i)}$ which comes from the Stirling approximation for $X_{\mathcal{I},\mathcal{J};t}$ as derived in Lemma 2.2.

Proposition 6.4.

$$I^{(2)}_O = \sum_{i_1=1}^{3}\sum_{i_2=1}^{3} I^{(2)}_{(i_1,i_2)} + O\Big(T^{\frac{3\vartheta}{2}+\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C} + \eta_C T^{\frac{3}{2}C-\frac{1}{2}+\frac{3\vartheta}{2}+\varepsilon}T_0^{-C}\Big). \tag{6.9}$$

where

$$I^{(2)}_{(i_1,i_2)} = \int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-\sum_{k\neq i_1}a_k-\sum_{k\neq i_2}b_k}\mathcal{Z}_{\mathcal{J}_{\bigcup_{k\neq i_1}\{a_k\}},\mathcal{J}_{\bigcup_{k\neq i_2}\{b_k\}}}(0)dt$$
$$-\frac{1}{2}\mathrm{Res}_{s=\frac{a_{i_1}+b_{i_2}}{2}}\mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s)\frac{G\Big(\frac{a_{i_1}+b_{i_2}}{2}\Big)}{\frac{a_{i_1}+b_{i_2}}{2}}\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-\sum_{k=1}^{3}(a_k+b_k)+\frac{3(a_{i_1}+b_{i_2})}{2}}dt$$
$$+\sum_{\substack{(k_1,k_2)\\ k_1\neq i_1, k_2\neq i_2}}\mathcal{U}_{(i_1,i_2);(k_1,k_2)} + O(T^{\frac{1}{2}+\varepsilon}) \tag{6.10}$$

and

(6.11)

$$\mathcal{U}_{(i_1,i_2);(k_1,k_2)} = \frac{1}{2}\prod_{j_1\neq i_1}\zeta(1+b_{i_1}-b_{j_1})\prod_{j_2\neq i_2}\zeta(1+a_{i_2}-a_{j_2})\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-a_{r(i_2,k_2)}-b_{r(i_1,k_1)}-\frac{b_{k_1}+a_{k_2}}{2}}dt$$
$$\times\Big(\prod_{\substack{l_1\neq i_1, l_2\neq i_2\\ (l_1,l_2)\neq(k_1,k_2)}}\zeta(1-b_{l_1}-a_{l_2}+b_{k_1}+a_{k_2})\Big)\zeta(1+b_{i_2}+a_{i_1}-b_{k_1}-a_{k_2})$$
$$\times\,\mathcal{C}_{-\mathcal{J},-\mathcal{I};\{-b_{i_2}\},\{-a_{i_1}\}}\big(\tfrac{b_{k_1}+a_{k_2}}{2}\big)\frac{G(\frac{b_{k_1}+a_{k_2}}{2})}{\frac{b_{k_1}+a_{k_2}}{2}}$$

where $r(i_1,k_1)$ and $r(i_2,k_2)$ are defined as follows:

(6.12) *Given distinct elements i,k of $\{1,2,3\}$, then $r=r(i,k)$ is the unique number r such that $\{1,2,3\}=\{i,k,r\}$.*

Remark. (i) Notice that the formulae for $I^{(2)}_{(i_1,i_2)}$ also contain extra unwanted residues $\mathcal{U}_{(i_1,i_2);(k_1,k_2)}$ that do not appear in (5.1). Fortunately, we shall establish that the $\mathcal{T}_{(i_1,i_2);(k_1,k_2)}$ and $\mathcal{U}_{(i_1,i_2);(k_1,k_2)}$ cancel each

other out.
(ii) It is possible to completely avoid having the $\mathcal{T}_{(i_1,i_2);(k_1,k_2)}$ and $\mathcal{U}_{(i_1,i_2);(k_1,k_2)}$ terms. This can be done by ensuring that $Q_{\mathcal{I},\mathcal{J}}(s)$ vanishes at some of these extra poles. We chose not to do this so that we could see the cancellation between various terms.

Proposition 6.5. *We have that*

$$\sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\sum_{\substack{(k_1,k_2)\\ k_1\neq i_1, k_2\neq i_2}}(\mathcal{T}_{(i_1,i_2);(k_1,k_2)}+\mathcal{U}_{(i_1,i_2);(k_1,k_2)})=0. \tag{6.13}$$

Proof of Proposition 5.1. Combining Propositions 6.3 and 6.4 we see that we get exactly the first three terms in (5.1) plus

$$\sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\sum_{\substack{(k_1,k_2)\\ k_1\neq i_1, k_2\neq i_2}}(\mathcal{T}_{(i_1,i_2);(k_1,k_2)}+\mathcal{U}_{(i_1,i_2);(k_1,k_2)})+O\Big(T^{\frac{3\vartheta}{2}+\varepsilon}\Big(\frac{T}{T_0}\Big)^{1+C}+\eta_C T^{\frac{3}{2}C-\frac{1}{2}+\frac{3\vartheta}{2}+\varepsilon}T_0^{-C}+T^{\frac{1}{2}+\varepsilon}\Big). \tag{6.14}$$

However, Proposition 6.5 shows that the sum in (6.14) equals 0. Thus we establish (5.1). □

Proof of Proposition 6.3. We shall focus on one of the nine terms. The one with $i_1=1$, $i_2=1$. We will obtain the result for other indices just by permuting them appropriately. We have that

$$I^{(1)}_{(1,1)}=\sum_{r\neq 0}\sum_{M,N}\frac{T}{\sqrt{MN}}\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathcal{I}}(1-a_1,\ell)G_{\mathcal{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}}$$
$$\int_{\max(0,r)}^{\infty}x^{-a_1}(x-r)^{-b_1}W\Big(\frac{x}{M}\Big)W\Big(\frac{x-r}{N}\Big)\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3x(x-r)}\Big)^s\int_{-\infty}^{\infty}\Big(1-\frac{r}{x}\Big)^{it}g(s,t)\omega(t)dtdsdx.$$

Since $W(x)=x^{-\frac{1}{2}}W_0(x)$ and W_0 satisfies (5.4) we have

$$\sum_{M}W\Big(\frac{x}{M}\Big)M^{-\frac{1}{2}}=x^{-\frac{1}{2}}\text{ and }\sum_{N}W\Big(\frac{x-r}{N}\Big)N^{-\frac{1}{2}}=(x-r)^{-\frac{1}{2}}.$$

Using these identities

$$I^{(1)}_{(1,1)}=T\sum_{r\neq 0}\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathcal{I}}(1-a_1,\ell)G_{\mathcal{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}}$$
$$\int_{\max(0,r)}^{\infty}x^{-\frac{1}{2}-a_1}(x-r)^{-\frac{1}{2}-b_1}\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3x(x-r)}\Big)^s\frac{1}{T}\int_{-\infty}^{\infty}\Big(1-\frac{r}{x}\Big)^{it}g(s,t)\omega(t)dtdsdx.$$

We write $I^{(1)}_{(1,1)}=I^{+}+I^{-1}$ where I^{+} is the sum over $r>0$ and I^{-} is the sum over $r<0$. We have that

$$I^{+}=T\sum_{r=1}^{\infty}\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathcal{I}}(1-a_1,\ell)G_{\mathcal{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}}$$
$$\times\int_{r}^{\infty}x^{-\frac{1}{2}-a_1}(x-r)^{-\frac{1}{2}-b_1}$$
$$\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3x(x-r)}\Big)^s\frac{1}{T}\int_{-\infty}^{\infty}\Big(1-\frac{r}{x}\Big)^{it}g(s,t)\omega(t)dtdsdx$$

and

$$I^- = T\sum_{r=1}^{\infty}\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathfrak{I}}(1-a_1,\ell)G_{\mathfrak{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}}$$
$$\times\int_0^\infty x^{-\frac12-a_1}(x+r)^{-\frac12-b_1}$$
$$\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3x(x+r)}\Big)^s\frac{1}{T}\int_{-\infty}^{\infty}\Big(1+\frac{r}{x}\Big)^{it}g(s,t)\omega(t)dtdsdx.$$

We let K^+ and K^- denote the triple integrals appearing in I^+ and I^-. In K^+ we make the change of variable $x = ry + r$ to obtain

$$K^+ = \int_r^\infty x^{-\frac12-a_1}(x-r)^{-\frac12-b_1}\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3x(x-r)}\Big)^s\frac{1}{T}\int_{-\infty}^{\infty}\Big(1-\frac{r}{x}\Big)^{it}g(s,t)\omega(t)dtdsdx$$
$$= r^{-a_1-b_1}\int_0^\infty(y+1)^{-\frac12-a_1}y^{-\frac12-b_1}\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3r^2(y+1)y}\Big)^s\frac{1}{T}\int_{-\infty}^{\infty}y^{it}(1+y)^{-it}g(s,t)\omega(t)dtdsdy.$$

Similarly, by the variable change $x = ry$

$$K^- = \int_0^\infty x^{-\frac12-a_1}(x+r)^{-\frac12-b_1}\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3x(x+r)}\Big)^s\frac{1}{T}\int_{-\infty}^{\infty}\Big(1+\frac{r}{x}\Big)^{it}g(s,t)\omega(t)dtdsdx$$
$$= r^{-a_1-b_1}\int_0^\infty y^{-\frac12-a_1}(1+y)^{-\frac12-b_1}\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3r^2y(y+1)}\Big)^s\frac{1}{T}\int_{-\infty}^{\infty}y^{-it}(1+y)^{it}g(s,t)\omega(t)dtdsdy.$$

Therefore

$$\begin{aligned}I^+ = T\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\sum_{r=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathfrak{I}}(1-a_1,\ell)G_{\mathfrak{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}r^{a_1+b_1}}\\ \int_0^\infty(x+1)^{-\frac12-a_1}x^{-\frac12-b_1}\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3r^2(x+1)x}\Big)^s\frac{1}{T}\int_{-\infty}^{\infty}x^{it}(1+x)^{-it}g(s,t)\omega(t)dtdsdx\end{aligned}\tag{6.15}$$

and

$$\begin{aligned}I^- = T\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\sum_{r=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathfrak{I}}(1-a_1,\ell)G_{\mathfrak{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}r^{a_1+b_1}}\\ \int_0^\infty x^{-\frac12-a_1}(1+x)^{-\frac12-b_1}\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3r^2x(x+1)}\Big)^s\frac{1}{T}\int_{-\infty}^{\infty}x^{-it}(1+x)^{it}g(s,t)\omega(t)dtdsdx.\end{aligned}\tag{6.16}$$

By the beta function identity $B(u,v)=\int_0^\infty x^{u-1}(1+x)^{-u-v}dx = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$ for $\Re(u),\Re(v)>0$,

$$\int_0^\infty(x+1)^{-\frac12-a_1-s-it}x^{-\frac12-b_1-s+it}dx = \frac{\Gamma(\frac12-b_1-s+it)\Gamma(a_1+b_1+2s)}{\Gamma(\frac12+a_1+s+it)}$$

and

$$\int_0^\infty(1+x)^{-\frac12-b_1-s+it}x^{-\frac12-a_1-s-it}dx = \frac{\Gamma(\frac12-a_1-s-it)\Gamma(a_1+b_1+2s)}{\Gamma(\frac12+b_1+s-it)}.$$

Inserting these identities in (6.15) and (6.16)

$$I^{+}=\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\sum_{r=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathcal{I}}(1-a_1,\ell)G_{\mathcal{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}r^{a_1+b_1}}$$
$$\int_{-\infty}^{\infty}\omega(t)\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}g(s,t)\Big(\frac{1}{\pi^3r^2}\Big)^s\frac{\Gamma(\frac12-b_1-s+it)\Gamma(a_1+b_1+2s)}{\Gamma(\frac12+a_1+s+it)}ds\,dt$$

and

$$I^{-}=\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\sum_{r=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathcal{I}}(1-a_1,\ell)G_{\mathcal{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}r^{a_1+b_1}}$$
$$\int_{-\infty}^{\infty}\omega(t)\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}g(s,t)\Big(\frac{1}{\pi^3r^2}\Big)^s\frac{\Gamma(\frac12-a_1-s-it)\Gamma(a_1+b_1+2s)}{\Gamma(\frac12+b_1+s-it)}ds\,dt.$$

However, we have the following consequence of Stirling's formula:
Let $0\le\Re(s)\le 1$, $t\in\mathbb{R}$, and $|a_1|,|b_1|\ll(\log T)^{-1}$. Then

$$\frac{\Gamma(\frac12-b_1-s+it)}{\Gamma(\frac12+a_1+s+it)}=t^{-a_1-b_1-2s}\exp(\tfrac{\pi i}{2}(-a_1-b_1-2s))(1+O(\tfrac{1+|s|^2}{t}))$$

and

$$\frac{\Gamma(\frac12-a_1-s-it)}{\Gamma(\frac12+b_1+s-it)}=t^{-a_1-b_1-2s}\exp(-\tfrac{\pi i}{2}(-a_1-b_1-2s))(1+O(\tfrac{1+|s|^2}{t})).$$

The proof of this is very similar to the proof of Lemma 2.2 (ii) and we leave it as an exercise. Thus

$$I^{+}=\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\sum_{r=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathcal{I}}(1-a_1,\ell)G_{\mathcal{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}r^{a_1+b_1}}\int_{-\infty}^{\infty}\omega(t)$$
$$\times\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}g(s,t)\Big(\frac{1}{\pi^3r^2}\Big)^s\Gamma(a_1+b_1+2s)t^{-a_1-b_1-2s}\exp\Big(\frac{\pi i}{2}(-a_1-b_1-2s)\Big)(1+O(\tfrac{1+|s|^2}{t}))ds\,dt.$$

and

$$I^{-}=\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\sum_{r=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathcal{I}}(1-a_1,\ell)G_{\mathcal{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}r^{a_1+b_1}}\int_{-\infty}^{\infty}\omega(t)$$
$$\times\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}g(s,t)\Big(\frac{1}{\pi^3r^2}\Big)^s\Gamma(a_1+b_1+2s)t^{-a_1-b_1-2s}\exp\Big(-\frac{\pi i}{2}(-a_1-b_1-2s)\Big)(1+O(\tfrac{1+|s|^2}{t}))ds\,dt.$$

We now combine I^+ and I^- to obtain

$$I^{(1)}_{(1,1)}=\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\sum_{r=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{c_\ell(r)G_{\mathcal{I}}(1-a_1,\ell)G_{\mathcal{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}r^{a_1+b_1}}$$
$$\int_{-\infty}^{\infty}\omega(t)\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}g(s,t)\Big(\frac{1}{\pi^3r^2}\Big)^s\Gamma(a_1+b_1+2s)t^{-a_1-b_1-2s}2\cos(\tfrac{\pi}{2}(a_1+b_1+2s))(1+O(\tfrac{1+|s|^2}{t}))ds\,dt.$$

We then move the s integral to the line $\Re(s)=1$ so that we may apply Proposition 6.1 (i). Moving to this line, swapping summation and integration order

$$I^{(1)}_{(1,1)}=\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\int_{-\infty}^{\infty}\omega(t)$$
$$\frac{1}{2\pi i}\int_{(1)}H_{\mathcal{I},\mathcal{J};a_1,b_1}(s)\frac{G(s)}{s}g(s,t)\pi^{-3s}\Gamma(a_1+b_1+2s)t^{-a_1-b_1-2s}2\cos(\tfrac{\pi}{2}(a_1+b_1+2s))(1+O(\tfrac{1+|s|^2}{t}))dt\,ds$$

where $H_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(s)$ is defined by (6.1). By Lemma 2.2 (ii), $g(s,t) = (\frac{t}{2})^s(1+O(|s|^2t^{-1}))$ and since $\Re(s)=1$ we may apply Proposition 6.1 to obtain

$$I^{(1)}_{(1,1)} = \prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\int_{-\infty}^{\infty}\omega(t)\times$$
$$\frac{1}{2\pi i}\int_{(1)}\zeta(a_1+b_1+2s)\Big(\prod_{\substack{(k_1,k_2)\\ k_1\neq 1,k_2\neq 1}}\zeta(1+a_{k_1}+b_{k_2}+2s)\Big)\mathcal{C}_{\mathcal{I},\mathcal{J};a_1,b_1}(s)\frac{G(s)}{s}\times$$
$$\Big(\frac{t}{2\pi}\Big)^{3s}\Gamma(a_1+b_1+2s)t^{-a_1-b_1-2s}2\cos(\tfrac{\pi}{2}(a_1+b_1+2s))(1+O(\tfrac{1+|s|^2}{t}))ds\,dt.$$

We now move the line back to $\Re(s)=\varepsilon$. It is at this point in the argument that we make use of the polynomial $Q_{\mathcal{I},\mathcal{J}}(s)$ which divides $G(s)$. Observe that the factor in brackets has poles at $\frac{1}{2}-\frac{a_{k_1}+b_{k_2}}{2}$ for $k_1\neq 1, k_2\neq 1$. However, these are cancelled by the zeros of $Q_{\mathcal{I},\mathcal{J}}(s)$. We now bound the contribution from the $O\Big(\frac{1+|s|^2}{t}\Big)$ term. We have $|\cos(\frac{\pi}{2}z)| \ll e^{\frac{\pi}{2}|\Im(z)|}$ for $|z|\geq 1$ and by Stirling's formula $|\Gamma(z)|\ll |y|^{x-\frac{1}{2}}e^{-\frac{\pi}{2}|y|}$ for $|y|\geq 1$. Combining these facts it follows that $|\Gamma(a_1+b_1+2s)\cos(\frac{\pi}{2}(a_1+b_1+2s))|\ll |u|^{3\varepsilon-\frac{1}{2}}$ where $s=\varepsilon+iu$. Also, since $|\zeta(\sigma+iu)|\ll |u|^{1/2}$ for $\sigma\geq 0$ and $\mathcal{C}_{\mathcal{I},\mathcal{J};a_1,b_1}(s)=O(1)$ in $\Re(s)\geq -\frac{1}{2}+\varepsilon$, we find that the error term contributes

$$\int_{-\infty}^{\infty}\omega(t)\int_{-\infty}^{\infty}|u|^{-1-A}t^{\varepsilon}|u|^{\frac{1}{2}}|u|^{3\varepsilon-\frac{1}{2}}\frac{1+|u|^2}{t}du\,dt \ll \int_{-\infty}^{\infty}\frac{\omega(t)}{t^{1-\varepsilon}}dt\ll T^{\varepsilon}.$$

It follows that $I^{(1)}_{(1,1)} = J^{(1)}_{(1,1)}+O(T^{\varepsilon})$ where

$$J^{(1)}_{(1,1)} = \prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\int_{-\infty}^{\infty}\omega(t)\frac{1}{2\pi i}\int_{(\varepsilon)}\zeta(a_1+b_1+2s)\Big(\prod_{\substack{(k_1,k_2)\\ k_1\neq 1,k_2\neq 1}}\zeta(1+a_{k_1}+b_{k_2}+2s)\Big)\times$$
$$\mathcal{C}_{\mathcal{I},\mathcal{J};a_1,b_1}(s)\frac{G(s)}{s}\Big(\frac{t}{2\pi}\Big)^{3s}\Gamma(a_1+b_1+2s)t^{-a_1-b_1-2s}2\cos(\tfrac{\pi}{2}(a_1+b_1+2s))ds\,dt.$$

By the functional equation in the unsymmetric form $\zeta(1-z)=2^{1-z}\pi^{-z}\cos(\frac{\pi z}{2})\Gamma(z)\zeta(z)$

$$J^{(1)}_{(1,1)} = \prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\int_{-\infty}^{\infty}\omega(t)\frac{1}{2\pi i}\int_{(\varepsilon)}\Big(\prod_{\substack{(k_1,k_2)\\ k_1\neq 1,k_2\neq 1}}\zeta(1+a_{k_1}+b_{k_2}+2s)\Big)\times$$
$$(2\pi)^{a_1+b_1+2s}\zeta(1-a_1-b_1-2s)\mathcal{C}_{\mathcal{I},\mathcal{J};a_1,b_1}(s)\frac{G(s)}{s}\Big(\frac{t}{2}\Big)^{3s}\pi^{-3s}t^{-a_1-b_1-2s}ds\,dt.$$

Further simplification yields

$$J^{(1)}_{(1,1)} = \prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-a_1-b_1}\frac{1}{2\pi i}\int_{(1)}\varphi(s)ds\,dt$$

where

$$\varphi(s) = \Big(\prod_{\substack{(k_1,k_2)\\ k_1\neq 1,k_2\neq 1}}\zeta(1+a_{k_1}+b_{k_2}+2s)\Big)\zeta(1-a_1-b_1-2s)\mathcal{C}_{\mathcal{I},\mathcal{J};a_1,b_1}(s)\frac{G(s)}{s}\Big(\frac{t}{2\pi}\Big)^{s}. \tag{6.17}$$

$\varphi(s)$ has poles at $s=0$, $s=-\frac{a_1+b_1}{2}$, and $-\frac{a_{k_1}+b_{k_2}}{2}$ for $k_1\neq 1$ and $k_2\neq 1$. We further evaluate $J^{(1)}_{(1,1)}$ by applying the residue theorem. We move the s contour left past $\Re(s)=0$, picking up residues at the various poles. Let

$$\mathcal{R}_1 = \text{Residue}(\varphi(s), s = 0), \tag{6.18}$$

$$\mathcal{R}_2 = \text{Residue}(\varphi(s), s = -\frac{a_1 + b_1}{2}), \tag{6.19}$$

$$\mathcal{R}_3 = \sum_{\substack{k_1 \neq 1 \\ k_2 \neq 1}} \text{Residue}(\varphi(s), s = -\frac{a_{k_1} + b_{k_2}}{2}). \tag{6.20}$$

By the residue theorem,

$$\frac{1}{2\pi i} \int_{(1)} \varphi(s) ds = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \frac{1}{2\pi i} \int_{(-\frac{1}{2}+\varepsilon)} \varphi(s) ds.$$

Observe that for $s = -\frac{1}{2} + \varepsilon + iu$ the zeta factors in $\varphi(s)$ are bounded by $(|u|+1)^{A_0}$ for some $A_0 > 0$ and $\mathcal{C}_{\mathcal{I},\mathcal{J};a_1,b_1}(s) \ll O(1)$ since $\Re(s) \geq -\frac{1}{2} + \varepsilon$. Therefore

$$\frac{1}{2\pi i} \int_{(c)} \varphi(s) ds \ll t^{-\frac{1}{2}+\varepsilon} \int_{-\infty}^{\infty} (|u|+1)^{A_0} |-\tfrac{1}{2} + \varepsilon + iu|^{-1} \min(1, |u|^{-A}) du \ll T^{-\frac{1}{2}+\varepsilon}$$

and

$$\int_{-\infty}^{\infty} \omega(t) \Big(\frac{t}{2\pi}\Big)^{-a_1-b_1} \frac{1}{2\pi i} \int_{(1)} \varphi(s) ds \, dt \ll T^{\frac{1}{2}+\varepsilon}.$$

In addition, as all the poles are simple we have the following residues:

$$\mathcal{R}_1 = \Big(\prod_{\substack{(k_1,k_2) \\ k_1 \neq 1, k_2 \neq 1}} \zeta(1 + a_{k_1} + b_{k_2}) \Big) \zeta(1 - a_1 - b_1) \mathcal{C}_{\mathcal{I},\mathcal{J};a_1,b_1}(0),$$

$$\mathcal{R}_2 = \Big(\prod_{\substack{(k_1,k_2) \\ k_1 \neq 1, k_2 \neq 1}} \zeta(1 + a_{k_1} + b_{k_2} - a_1 - b_1) \Big) \mathcal{C}_{\mathcal{I},\mathcal{J};a_1,b_1}(-\tfrac{a_1+b_1}{2}) \frac{G(-\frac{a_1+b_1}{2})}{-\frac{a_1+b_1}{2}} \Big(\frac{t}{2\pi}\Big)^{-\frac{a_1+b_1}{2}},$$

and the residue at $s = -\frac{a_{k_1}+b_{k_2}}{2}$ equals

$$\mathcal{R}_3 = \sum_{\substack{k_1 \neq 1 \\ k_2 \neq 1}} \prod_{\substack{l_1 \neq 1, l_2 \neq 1 \\ (l_1,l_2) \neq (k_1,k_2)}} \zeta(1 + a_{l_1} + b_{l_2} - a_{k_1} - b_{k_2}) \zeta(1 - a_1 - b_1 + a_{k_1} + b_{k_2}) \mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\tfrac{a_{k_1}+b_{k_2}}{2})$$
$$\times \frac{G(-\frac{a_{k_1}+b_{k_2}}{2})}{-\frac{a_{k_1}+b_{k_2}}{2}} \Big(\frac{t}{2\pi}\Big)^{-\frac{a_{k_1}+b_{k_2}}{2}}.$$

It follows that

$$I_{1,1}^{(1)} = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + O(T^{\frac{1}{2}+\varepsilon}) \tag{6.21}$$

where

$$\begin{aligned} \mathcal{S}_1 &= \prod_{j_1 \neq 1} \zeta(1 - a_1 + a_{j_1}) \prod_{j_2 \neq 1} \zeta(1 - b_1 + b_{j_2}) \int_{-\infty}^{\infty} \omega(t) \Big(\frac{t}{2\pi}\Big)^{-a_1-b_1} \\ &\quad \times \Big(\prod_{\substack{(k_1,k_2) \\ k_1 \neq 1, k_2 \neq 1}} \zeta(1 + a_{k_1} + b_{k_2}) \Big) \zeta(1 - a_1 - b_1) \mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(0) dt \\ &= \zeta_{\mathcal{I}_{\{a_1\}},\mathcal{J}_{\{b_1\}}}(0) \int_{-\infty}^{\infty} \omega(t) \Big(\frac{t}{2\pi}\Big)^{-a_1-b_1} \mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(0) dt, \end{aligned} \tag{6.22}$$

$$\mathcal{S}_2 = -\frac{1}{2}\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-\frac{3(a_1+b_1)}{2}}$$

$$\times\prod_{\substack{(k_1,k_2)\\ k_1\neq 1,k_2\neq 1}}\zeta(1+a_{k_1}+b_{k_2}-a_1-b_1)\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\tfrac{a_1+b_1}{2})\frac{G(-\frac{a_1+b_1}{2})}{-\frac{a_1+b_1}{2}}dt, \tag{6.23}$$

(6.24)

$$\mathcal{S}_3 = \frac{1}{2}\prod_{j_1\neq 1}\zeta(1-a_1+a_{j_1})\prod_{j_2\neq 1}\zeta(1-b_1+b_{j_2})\sum_{\substack{(k_1,k_2)\\ k_1\neq 1,k_2\neq 1}}\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-a_1-b_1-\frac{a_{k_1}+b_{k_2}}{2}}\times$$

$$\Big(\prod_{\substack{l_1\neq 1,l_2\neq 1\\ (l_1,l_2)\neq(k_1,k_2)}}\zeta(1+a_{l_1}+b_{l_2}-a_{k_1}-b_{k_2})\Big)\zeta(1-a_1-b_1+a_{k_2}+b_{k_2})\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\tfrac{a_{k_1}+b_{k_2}}{2})\frac{G(-\frac{a_{k_1}+b_{k_2}}{2})}{-\frac{a_{k_1}+b_{k_2}}{2}}dt.$$

We now provide further simplification of $\mathcal{S}_1$ and $\mathcal{S}_2$. We would like to prove that

$$\mathcal{S}_1 = \int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-a_1-b_1}\mathcal{Z}_{\mathcal{I}_{\{a_1\}},\mathcal{J}_{\{b_1\}}}(0)dt. \tag{6.25}$$

Glancing at (6.22), this follows from the identity $\mathcal{A}_{\mathcal{I}_{\{a_1\}},\mathcal{J}_{\{b_1\}}}(0) = \mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(0)$ which is Proposition 6.2 (i). Next, we show that

$$\mathcal{S}_2 = -\mathrm{Res}_{s=\frac{-a_1-b_1}{2}}\mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s)\frac{G\Big(\frac{-a_1-b_1}{2}\Big)}{\frac{-a_1-b_1}{2}}\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-\frac{3}{2}(a_1+b_1)}dt. \tag{6.26}$$

First observe that

$$\mathrm{Res}_{s=\frac{-a_1-b_1}{2}}\mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s) = \frac{1}{2}\Big(\prod_{\substack{(i,j)\\ (i,j)\neq(1,1)}}\zeta(1+a_i+b_j-a_1-b_1)\Big)\mathcal{A}_{\mathcal{I},\mathcal{J}}(-a_1-b_1).$$

Thus, in order to prove (6.26), it suffices to prove that $\mathcal{A}_{\mathcal{I},\mathcal{J}}(-a_1-b_1) = \mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\frac{a_1+b_1}{2})$ which is Proposition 6.2 (ii). From (6.21), (6.25), (6.26), and (6.24) we arrive at (6.7) in the case that $i_1 = i_2 = 1$. The case of general i_1, i_2, follows from a simple permutation of variables. ☐

Proposition 6.4 may be proven by a calculation analogous to the proof of Proposition 6.3.

Proof of Proposition 6.4. Rather than repeat the proof of Proposition 6.3 line by line we just mention the differences in the calculation. First, we have the factor $X_{\mathcal{I},\mathcal{J};t}$ present which leads to an extra factor of

$$X_{\mathcal{I},\mathcal{J};t} \sim (t/2\pi)^{-\sum_{k=1}^{3}(a_k+b_k)} \tag{6.27}$$

and second we have $\mathcal{I}$ is replaced by $-\mathcal{J}$ and $\mathcal{J}$ is replaced by $-\mathcal{I}$ or

$$a_i \to -b_i \text{ and } b_i \to -a_i \text{ for } i = 1,2,3. \tag{6.28}$$

We could repeat exactly our proof of Proposition 6.3 and obtain the result. However, these differences mean that the formula in Proposition 6.4 can be obtained by inserting the factor $(t/2\pi)^{-\sum_{k=1}^{3}(a_k+b_k)}$ and permuting the variables as in (6.28). We shall obtain the formula for $I^{(2)}_{(1,1)}$ from $I^{(1)}_{(1,1)}$ by doing this. The first term in $I^{(2)}_{(1,1)}$ is

$$\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-\sum_{k=1}^{3}(a_k+b_k)+a_1+b_1}\mathcal{Z}_{-\mathcal{J}_{\{-b_1\}},\mathcal{I}_{\{-a_1\}}}(0)dt. \tag{6.29}$$

Note that

$$(-\mathcal{J}_{\{-b_1\}}; -\mathcal{I}_{\{-a_1\}}) = \{a_1, -b_2, -b_3; b_1, -a_2, -a_3\} = (\mathcal{I}_{\{a_2,a_3\}}; \mathcal{J}_{\{b_2,b_3\}})$$

and thus this equals

$$\int_{-\infty}^{\infty} \omega(t)\Big(\frac{t}{2\pi}\Big)^{-a_2-a_3-b_2-b_3} \mathcal{Z}_{\mathcal{I}_{\{a_2,a_3\}},\mathcal{J}_{\{b_2,b_3\}}}(0)dt.$$

Similarly, we find the second term of $I^{(2)}_{(1,1)}$ is

$$-\frac{1}{2}\mathrm{Res}_{s=\frac{a_1+b_1}{2}} \mathcal{Z}_{\mathcal{I},\mathcal{J}}(2s) \frac{G\Big(\frac{a_1+b_1}{2}\Big)}{\frac{a_1+b_1}{2}} \int_{-\infty}^{\infty} \omega(t)\Big(\frac{t}{2\pi}\Big)^{-\sum_{k=1}^{3}(a_k+b_k)+\frac{3(a_1+b_1)}{2}} dt \tag{6.30}$$

and the third term is

$$\begin{aligned}
&+\frac{1}{2}\prod_{j_1\neq 1}\zeta(1+b_1-b_{j_1})\prod_{j_2\neq 1}\zeta(1+a_1-a_{j_2}) \sum_{\substack{(k_1,k_2)\\ k_1\neq 1, k_2\neq 1}} \int_{-\infty}^{\infty} \omega(t)\Big(\frac{t}{2\pi}\Big)^{-\sum_{k=1}^{3}(a_k+b_k)+a_1+b_1+\frac{b_{k_1}+a_{k_2}}{2}} dt \\
&\times \Big(\prod_{\substack{l_1\neq 1, l_2\neq 1\\ (l_1,l_2)\neq(k_1,k_2)}} \zeta(1-b_{l_1}-a_{l_2}+b_{k_1}+a_{k_2})\Big)\zeta(1+b_1+a_1-b_{k_2}-a_{k_2}) \\
&\times \mathcal{C}_{-\mathcal{J},-\mathcal{I};\{-b_1\},\{-a_1\}}\big(\tfrac{b_{k_1}+a_{k_2}}{2}\big)\frac{G(\frac{b_{k_1}+a_{k_2}}{2})}{\frac{b_{k_1}+a_{k_2}}{2}}.
\end{aligned}$$

Thus we see that the above expression simplifies to

$$\begin{aligned}
&+\frac{1}{2}\prod_{j_1\neq 1}\zeta(1+b_1-b_{j_1})\prod_{j_2\neq 1}\zeta(1+a_1-a_{j_2}) \sum_{\substack{(k_1,k_2)\\ k_1\neq 1, k_2\neq 1}} \int_{-\infty}^{\infty} \omega(t)\Big(\frac{t}{2\pi}\Big)^{-a_{r_2}-b_{r_1}-\frac{b_{k_1}+a_{k_2}}{2}} dt \\
&\times \Big(\prod_{\substack{l_1\neq 1, l_2\neq 1\\ (l_1,l_2)\neq(k_1,k_2)}} \zeta(1-b_{l_1}-a_{l_2}+b_{k_1}+a_{k_2})\Big)\zeta(1+b_1+a_1-b_{k_1}-a_{k_2}) \\
&\times \mathcal{C}_{-\mathcal{J},-\mathcal{I};\{-b_1\},\{-a_1\}}\big(\tfrac{b_{k_1}+a_{k_2}}{2}\big)\frac{G(\frac{b_{k_1}+a_{k_2}}{2})}{\frac{b_{k_1}+a_{k_2}}{2}}.
\end{aligned} \tag{6.31}$$

where we recall that $r_1 = r_1(1, k_1)$ and $r_2 = r_2(1, k_2)$ are defined by (6.12). Hence, we find that $I^{(2)}_{(1,1)}$ equals the sum of (6.29), (6.30), and (6.31). This is precisely (6.10) in the case $(i_1, i_2) = (1, 1)$. The general case follows from the permutation $1 \to i_1$ and $1 \to i_2$. □

Proof of Lemma 6.5. Recall that we are trying to prove that

$$\sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\sum_{\substack{(k_1,k_2)\\ k_1\neq i_1, k_2\neq i_2}} (\mathcal{T}_{(i_1,i_2);(k_1,k_2)} + \mathcal{U}_{(i_1,i_2);(k_1,k_2)}) = 0. \tag{6.32}$$

We aim to prove this by matching terms in the two triple sums. First we show that

$$\mathcal{T}_{(1,1);(2,2)} + \mathcal{U}_{(3,3);(2,2)} = 0. \tag{6.33}$$

We begin with a few observations. Note that

(6.34)

$$\mathcal{T}_{(1,1);(2,2)} = \frac{1}{2}\zeta(1-a_1+a_2)\zeta(1-a_1+a_3)\zeta(1-b_1+b_2)\zeta(1-b_1+b_3)\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-a_1-b_1-\frac{a_2+b_2}{2}}dt$$
$$\times\,\zeta(1+a_3+b_2-a_2-b_2)\zeta(1+a_2+b_3-a_2-b_2)\zeta(1+a_3+b_3-a_2-b_2)\zeta(1-a_1-b_1+a_2+b_2)$$
$$\times\,\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\tfrac{a_2+b_2}{2})\frac{G(-\frac{a_2+b_2}{2})}{-\frac{a_2+b_2}{2}}$$
$$= -\frac{1}{2}\zeta(1-a_1+a_2)\zeta(1-a_1+a_3)\zeta(1-b_1+b_2)\zeta(1-b_1+b_3)\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-a_1-b_1-\frac{a_2+b_2}{2}}dt$$
$$\times\,\zeta(1+a_3-a_2)\zeta(1+b_3-b_2)\zeta(1+a_3+b_3-a_2-b_2)\zeta(1-a_1-b_1+a_2+b_2)$$
$$\times\,\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\tfrac{a_2+b_2}{2})\frac{G(\frac{a_2+b_2}{2})}{\frac{a_2+b_2}{2}},$$

since G is even. We now try to identify a term which will cancel with this. We shall look in the terms coming from the second half of the approximate functional equation. We guess the correct term arises from $I^{(2)}_{(3,3)}$ and is $\mathcal{U}_{(3,3);(2,2)}$. Note that $r_1(3,2) = r_2(3,2) = 1$ so that

(6.35)

$$\mathcal{U}_{(3,3);(2,2)} = \frac{1}{2}\zeta(1+b_3-b_1)\zeta(1+b_3-b_2)\zeta(1+a_3-a_1)\zeta(1+a_3-a_2)\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-a_1+b_1-\frac{b_2+a_2}{2}}dt$$
$$\times\,\zeta(1-b_1-a_1+b_2+a_2)\zeta(1-b_1-a_2+b_2+a_2)\zeta(1-b_2-a_1+b_2+a_2)\zeta(1+b_3+a_3-b_2-a_2)$$
$$\times\,\mathcal{C}_{-\mathcal{J},-\mathcal{I};\{-b_3\},\{-a_3\}}(\tfrac{b_2+a_2}{2})\frac{G(\frac{b_2+a_2}{2})}{\frac{b_2+a_2}{2}}$$
$$= \frac{1}{2}\zeta(1+b_3-b_1)\zeta(1+b_3-b_2)\zeta(1+a_3-a_1)\zeta(1+a_3-a_2)\int_{-\infty}^{\infty}\omega(t)\Big(\frac{t}{2\pi}\Big)^{-a_1+b_1-\frac{b_2+a_2}{2}}dt$$
$$\times\,\zeta(1-b_1-a_1+b_2+a_2)\zeta(1-b_1+b_2)\zeta(1-a_1+a_2)\zeta(1+b_3+a_3-b_2-a_2)$$
$$\times\,\mathcal{C}_{-\mathcal{J},-\mathcal{I};\{-b_3\},\{-a_3\}}(\tfrac{b_2+a_2}{2})\frac{G(\frac{b_2+a_2}{2})}{\frac{b_2+a_2}{2}}.$$

Observe that the two expressions we are considering are negatives of each other and add to zero if

$$\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\tfrac{a_2+b_2}{2}) = \mathcal{C}_{-\mathcal{J},-\mathcal{I};\{-b_3\},\{-a_3\}}(\tfrac{b_2+a_2}{2}). \tag{6.36}$$

However, this identity is Proposition (6.2) (iii). Thus this establishes (6.33). More generally, we can show that for $(i_1, i_2) \in \{1,2,3\}^2$ and $(k_1,k_2) \in \{1,2,3\}^2$ such that $k_1 \neq i_1$ and $k_2 \neq i_2$ that

$$\mathcal{T}_{(i_1,i_2);(k_1,k_2)} + \mathcal{U}_{(r_1,r_2);(k_1,k_2)} = 0 \tag{6.37}$$

where we recall that $r_1 = r_1(i_1,k_1)$ and $r_2 = r_2(i_2,k_2)$ are defined by (6.12). By an analogous argument this would be true if

$$\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_{i_1}\},\{b_{i_1}\}}(-\tfrac{a_{k_1}+b_{k_2}}{2}) = \mathcal{C}_{-\mathcal{J},-\mathcal{I};\{-b_{r_1}\},\{-a_{r_2}\}}(\tfrac{b_{k_1}+a_{k_2}}{2}).$$

This follows from Proposition (6.2) (iii) by a permutation of variables. Finally, summing (6.37) over i_1, i_2, k_1, k_2 leads to (6.32). □

7. Appendix 1: Proof of Propositions 6.1, 6.2

We now establish Proposition 6.1.

Lemma 7.1. *Let $k \in \mathbb{N}$, $I = \{1, \dots, k\}$, and $X = \{x_1, x_2, \dots, x_k\}$ are distinct complex numbers. For p prime and $\alpha \geq 0$*

$$g_X(s, p^\alpha) = (1 - p^{-s-x_1}) \cdots (1 - p^{-s-x_k}) \sum_{i=1}^{k} \frac{p^{-x_i \alpha}}{1 - p^{-x_i - s}} \prod_{\ell \in I \setminus \{i\}} (1 - p^{x_i - x_\ell})^{-1}. \tag{7.1}$$

Proof. We begin by recalling that

$$g_X(s, p^j) = \frac{\sum_{j=0}^{\infty} \frac{\sigma_X(p^{\alpha+j})}{p^{js}}}{\sum_{j=0}^{\infty} \frac{\sigma_X(p^j)}{p^{js}}}. \tag{7.2}$$

We now find an expression for $\sigma_X(p^j)$ for $j \geq 0$. By multiplicativity

$$\zeta(s + x_1) \cdots \zeta(s + x_k) = \sum_{n=1}^{\infty} \sigma_X(n) n^{-s} = \prod_p \sum_{j=0}^{\infty} \sigma_X(p^j) p^{-js}.$$

On the other hand

$$\zeta(s + x_1) \cdots \zeta(s + x_k) = \prod_p (1 - p^{-s-x_1})^{-1} \cdots (1 - p^{-s-x_k})^{-1}$$

and it follows that

$$\sum_{j=0}^{\infty} \sigma_X(p^j) p^{-js} = (1 - p^{-s-x_1})^{-1} \cdots (1 - p^{-s-x_k})^{-1}. \tag{7.3}$$

By partial fractions,

$$(1 - p^{-s-x_1})^{-1} \cdots (1 - p^{-s-x_k})^{-1} = \sum_{i=1}^{k} (1 - p^{-s-x_i})^{-1} \prod_{\ell \in I \setminus \{i\}} (1 - p^{x_i - x_\ell})^{-1} \tag{7.4}$$

Expanding the right hand side by the geometric series

$$(1 - p^{-s-x_1})^{-1} \cdots (1 - p^{-s-x_k})^{-1} = \sum_{i=1}^{k} \prod_{\ell \in I \setminus \{i\}} (1 - p^{x_i - x_l})^{-1} \sum_{j=0}^{\infty} p^{-x_i j} p^{-js}. \tag{7.5}$$

From (7.3) and (7.5) we deduce for $j \geq 0$

$$\sigma_X(p^j) = \sum_{i=1}^{k} p^{-x_i j} \prod_{\ell \in I \setminus \{i\}} (1 - p^{x_i - x_\ell})^{-1}.$$

Hence for $\alpha \geq 1$

$$\sum_{j=0}^{\alpha-1} \sigma_X(p^j) p^{-js} = \sum_{i=1}^{k} \frac{1 - (p^{-x_i - s})^\alpha}{1 - p^{-x_i - s}} \prod_{\ell \in I \setminus \{i\}} (1 - p^{x_i - x_\ell})^{-1}.$$

Thus

$$
\begin{aligned}
\sum_{j=0}^{\infty}\sigma_X(p^{j+\alpha})p^{-js} &= p^{\alpha s}\sum_{j\ge\alpha}\sigma_X(p^j)p^{-js}\\
&= p^{\alpha s}\Big(\sum_{j\ge 0}\sigma_X(p^j)p^{-js} - \sum_{j=0}^{\alpha-1}\sigma_X(p^j)p^{-js}\Big)\\
&= p^{\alpha s}\sum_{i=1}^{k}\frac{(p^{-x_i-s})^{\alpha}}{1-p^{-x_i-s}}\prod_{\ell\in I\setminus\{i\}}(1-p^{x_i-x_l})^{-1}\\
&= \sum_{i=1}^{k}\frac{p^{-x_i\alpha}}{1-p^{-x_i-s}}\prod_{\ell\in I\setminus\{i\}}(1-p^{x_i-x_\ell})^{-1}.
\end{aligned}
\tag{7.6}
$$

By (7.2), (7.3), and (7.6) we obtain (7.1) for $\alpha \ge 1$. In the case that $\alpha = 0$, we observe that the left hand side equals 1 since $g_X(s,n)$ is multiplicative and the right hand side also equals 1 by (7.4). $\square$

Lemma 7.2. *Let $k\in\mathbb{N}$, $I=\{1,\dots,k\}$, and $X=\{x_1,x_2,\dots,x_k\}$ be distinct complex numbers. For p prime and $j\ge 1$*

$$
G_X(s,p^j) = (1-p^{-s-x_1})\cdots(1-p^{-s-x_k})\frac{1}{p-1}\sum_{i=1}^{k}\frac{p^{1-x_ij}-p^{s-x_i(j-1)}}{1-p^{-x_i-s}}\prod_{\ell\in I\setminus\{i\}}(1-p^{x_i-x_\ell})^{-1}. \tag{7.7}
$$

Proof. By definition (1.24) it follows that

$$
\begin{aligned}
G_X(s,p^j) &= \sum_{d\mid p^j}\frac{\mu(d)d^s}{\phi(d)}\sum_{e\mid d}\frac{\mu(e)}{e^s}g_X\Big(s,\frac{p^je}{d}\Big) = g_X(s,p^j) - \frac{p^s}{p-1}g_X(s,p^{j-1}) + \frac{1}{\phi(p)}g_X(s,p^j)\\
&= \frac{p}{p-1}g_X(s,p^j) - \frac{p^s}{p-1}g_X(s,p^{j-1}).
\end{aligned}
$$

Inserting (7.1) in the last expression with $\alpha = j$ and $\alpha = j-1$ yields (7.7). $\square$

Observe that we may apply the preceding result in the special case $X = \mathcal{I} = \{a_1,a_2,a_3\}$.

Lemma 7.3. *Let $\mathcal{I}=\{a_1,a_2,a_3\}$ be distinct complex numbers, p a prime, and $j\ge 1$. Then*

$$
G_{\mathcal{I}}(1-a_1,p^j) = p^{-a_2j}\frac{1-p^{-1+a_1-a_3}}{1-p^{a_2-a_3}} + p^{-a_3j}\frac{1-p^{-1+a_1-a_2}}{1-p^{a_3-a_2}} \tag{7.8}
$$

and, in particular,

$$
G_{\mathcal{I}}(1-a_1,p) = p^{-a_2}+p^{-a_3}-p^{-1+a_1-a_2-a_3}.
$$

Proof. By Lemma 7.2 it follows that

$$
\begin{aligned}
G_{\mathcal{I}}(1-a_1,p^j) &= (1-p^{-1})(1-p^{-1+a_1-a_2})(1-p^{-1+a_1-a_3})\frac{1}{p-1}\\
&\quad\times\sum_{i=1}^{3}\frac{p^{1-a_ij}-p^{(1-a_1)-a_i(j-1)}}{1-p^{-a_i-(1-a_1)}}\prod_{\ell\in I\setminus\{i\}}(1-p^{a_i-a_\ell})^{-1}.
\end{aligned}
$$

Note that if $i=1$, then $p^{1-a_1j}-p^{(1-a_1)-a_1(j-1)}=0$ and $(1-p^{-1})(p-1)^{-1}=p^{-1}$. Therefore

$$
\begin{aligned}
&G_{\mathcal{I}}(1-a_1,p^j) = (1-p^{-1+a_1-a_2})(1-p^{-1+a_1-a_3})\\
&\times\Big(\frac{p^{-a_1j}(1-p^{a_2-a_1})}{(1-p^{-1+a_1-a_2})(1-p^{a_2-a_1})(1-p^{a_2-a_3})} + \frac{p^{-a_2j}(1-p^{a_3-a_1})}{(1-p^{-1+a_1-a_3})(1-p^{a_3-a_1})(1-p^{a_3-a_2})}\Big).
\end{aligned}
$$

Simplifying this yields (7.8). $\square$

With Lemma 7.3 in hand we can proceed with the proof of Proposition (6.1).

Proof of Proposition (6.1). (i) By $c_\ell(r) = \sum_{d|(\ell,r)} d\mu(\frac{\ell}{d})$ we have

$$H_{\mathfrak{I},\mathfrak{J};\{a_1\},\{b_1\}}(s) = \sum_{r=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{G_{\mathfrak{I}}(1-a_1,\ell)G_{\mathfrak{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}r^{a_1+b_1+2s}} \sum_{d|(l,r)} d\mu(\tfrac{\ell}{d})$$
$$= \sum_{\ell=1}^{\infty}\alpha_\ell \sum_{r=1}^{\infty}\frac{1}{r^c}\sum_{d|l,d|r} d\mu(\tfrac{\ell}{d})$$

where $\alpha_\ell = \frac{G_{\mathfrak{I}}(1-a_1,\ell)G_{\mathfrak{J}}(1-b_1,\ell)}{\ell^{2-a_1-b_1}}$ and $c = a_1+b_1+2s$. Thus

$$H_{\mathfrak{I},\mathfrak{J};\{a_1\},\{b_1\}}(s) = \sum_{\ell=1}^{\infty}\alpha_\ell\sum_{d|\ell} d\mu(\tfrac{\ell}{d})\sum_{r=1,d|r}\frac{1}{r^c} = \sum_{\ell=1}^{\infty}\alpha_\ell\sum_{d|l}\frac{d\mu(\frac{\ell}{d})}{d^c}\zeta(c)$$
$$= \zeta(c)\sum_{\ell=1}^{\infty}\alpha_\ell\sum_{d|\ell} d^{1-c}\mu(\tfrac{\ell}{d}) = \zeta(c)\sum_{\ell=1}^{\infty}\alpha_\ell \ell^{1-c}\sum_{d|\ell}\frac{\mu(d)}{d^{1-c}}.$$

For p prime and $j \geq 1$ we have $\sum_{d|p^j}\frac{\mu(d)}{d^{1-c}} = 1 - \frac{1}{p^{1-c}}$. By multiplicativity

$$\begin{aligned}\sum_{\ell=1}^{\infty}\alpha_\ell\ell^{1-c}\sum_{d|\ell}\frac{\mu(d)}{d^{1-c}} &= \prod_p\Big(1+\sum_{j=1}^{\infty}\frac{G_{\mathfrak{I}}(1-a_1,p^j)G_{\mathfrak{J}}(1-b_1,p^j)}{(p^j)^{2-a_1-b_1}}(p^j)^{1-a_1-b_1-2s}(1-p^{a_1-b_1+2s-1})\Big)\\ &= \prod_p\Big(1+\sum_{j=1}^{\infty}\frac{G_{\mathfrak{I}}(1-a_1,p^j)G_{\mathfrak{J}}(1-b_1,p^j)}{(p^j)^{1+2s}}(1-p^{a_1+b_1+2s-1})\Big).\end{aligned}\tag{7.9}$$

We shall begin by determining the first two terms of the last expression in brackets. By Lemma 7.3

$$G_{\mathfrak{I}}(1-a_1,p) = p^{-a_2}+p^{-a_3}-p^{-1+a_1-a_2-a_3} \text{ and } G_{\mathfrak{J}}(1-b_1,p) = p^{-b_2}+p^{-b_3}-p^{-1+b_1-b_2-b_3}.$$

Therefore the first two terms equal

$$\begin{aligned}&1+(p^{-a_2}+p^{-a_3}-p^{-1+a_1-a_2-a_3})(p^{-b_2}+p^{-b_3}-p^{-1+b_1-b_2-b_3})(p^{-1-2s}-p^{a_1+b_1-2})\\ &= 1+p^{-1-a_2-b_2-2s}+p^{1-a_2-b_3-2s}+p^{-1-a_3-b_2-2s}+p^{-1-a_3-b_3-2s}+O(p^{-2-2\sigma}+p^{-2}).\end{aligned}\tag{7.10}$$

It follows that the sum over ℓ in (7.9) equals

$$\zeta(1+a_2+b_2+2s)\zeta(1+a_2+b_3+2s)\zeta(1+a_3+b_2+2s)\zeta(1+a_3+b_3+2s)\mathcal{C}_{\mathfrak{I},\mathfrak{J};\{a_1\},\{b_1\}}(s)$$

where

$$\mathcal{C}_{\mathfrak{I},\mathfrak{J};\{a_1\},\{b_1\}}(s) = \prod_p \mathcal{C}_{\mathfrak{I},\mathfrak{J};\{a_1\},\{b_1\}}(p;s)\tag{7.11}$$

and

$$\begin{aligned}\mathcal{C}_{\mathfrak{I},\mathfrak{J};\{a_1\},\{b_1\}}(p;s) &= \Big(1+\sum_{j=1}^{\infty}\frac{G_{\mathfrak{I}}(1-a_1,p^j)G_{\mathfrak{J}}(1-b_1,p^j)}{(p^j)^{1+2s}}(1-p^{a_1+b_1+2s-1})\Big)\times\\ &\Big(1-\frac{1}{p^{1+a_2+b_2+2s)}}\Big)\Big(1-\frac{1}{p^{1+a_2+b_3+2s)}}\Big)\Big(1-\frac{1}{p^{1+a_3+b_2+2s)}}\Big)\Big(1-\frac{1}{p^{1+a_3+b_3+2s)}}\Big).\end{aligned}\tag{7.12}$$

Hence

$$H_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(s) = \zeta(a_1+b_1+2s)$$
$$\zeta(1+a_2+b_2+2s)\zeta(1+a_2+b_3+2s)\zeta(1+a_3+b_2+2s)\zeta(1+a_3+b_3+2s)\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(s).$$

We now demonstrate (6.5). It will be convenient to set $r(p) = p^{-a_2-b_2} + p^{-a_2-b_3} + p^{-a_3-b_2} + p^{-a_3-b_3}$. Note that $G_{\mathcal{I}}(1-a_1,p^j), G_{\mathcal{J}}(1-b_1,p^j) \ll p^{j\varepsilon}$. Therefore

$$\sum_{j=2}^{\infty} \frac{G_{\mathcal{I}}(1-a_1,p^j)G_{\mathcal{J}}(1-b_1,p^j)}{(p^j)^{1+2s}}(1-p^{a_1+b_1+2s-1}) \ll \sum_{j=2}^{\infty} \frac{1}{(p^{1+2\sigma-\varepsilon})^j} + \sum_{j=2}^{\infty} \frac{p^{|a_1|+|b_1|-2}}{p^{(j-1)(1+2\sigma)}}$$
$$\ll \frac{1}{p^{2+4\sigma-2\varepsilon}} + \frac{1}{p^{3+2\sigma-2\varepsilon}}.$$

It follows from (7.10) and the last inequality that

$$1 + \sum_{j=1}^{\infty} \frac{G_{\mathcal{I}}(1-a_1,p^j)G_{\mathcal{J}}(1-b_1,p^j)}{(p^j)^{1+2s}}(1-p^{a_1+b_1+2s-1}) = 1 + \frac{r(p)}{p^{1+2s}} + O(p^{-2-2\sigma}+p^{-2}). \tag{7.13}$$

On the other hand, by multiplying out

$$\begin{aligned}&\Big(1-\frac{1}{p^{1+a_2+b_2+2s)}}\Big)\Big(1-\frac{1}{p^{1+a_2+b_3+2s)}}\Big)\Big(1-\frac{1}{p^{1+a_3+b_2+2s)}}\Big)\Big(1-\frac{1}{p^{1+a_3+b_3+2s)}}\Big)\\ &= 1 - \frac{r(p)}{p^{1+2s}} + O(p^{-2-4\sigma}).\end{aligned} \tag{7.14}$$

Multiplying (7.13) and (7.14) we obtain

$$\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(p;s) = 1 + O\Big(\frac{1}{p^{2+2\sigma}}\Big). \tag{7.15}$$

The next step is to derive an explicit formula for $\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(s)$, namely (6.3). By (7.8) It follows that

$$\begin{aligned}\frac{G_{\mathcal{I}}(1-a_1,p^j)G_{\mathcal{J}}(1-b_1,p^j)}{(p^j)^{1+2s}} = \Bigg(&p^{-(1+a_2+b_2+2s)j}\frac{1-p^{-1+a_1-a_3}}{1-p^{a_2-a_3}}\frac{1-p^{-1+b_1-b_3}}{1-p^{b_2-b_3}}\\ &+ p^{-(1+a_2+b_3+2s)j}\frac{1-p^{-1+a_1-a_3}}{1-p^{a_2-a_3}}\frac{1-p^{-1+b_1-b_2}}{1-p^{b_3-b_2}}\\ &+ p^{-(1+a_3+b_2+2s)j}\frac{1-p^{-1+a_1-a_2}}{1-p^{a_3-a_2}}\frac{1-p^{-1+b_1-b_3}}{1-p^{b_2-b_3}}\\ &+ p^{-(1+a_3+b_3+2s)j}\frac{1-p^{-1+a_1-a_2}}{1-p^{a_3-a_2}}\frac{1-p^{-1+b_1-b_2}}{1-p^{b_3-b_2}}\Bigg).\end{aligned}$$

Since $\sum_{j=1}^{\infty} p^{-j\kappa} = \frac{p^{-\kappa}}{1-p^{-\kappa}}$, it follows that

$$
\begin{aligned}
&\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(p;s) = \\
&1 + \Big(\frac{p^{-(1+a_2+b_2+2s)}}{1-p^{-(1+a_2+b_2+2s)}} \frac{1-p^{-1+a_1-a_3}}{1-p^{a_2-a_3}} \frac{1-p^{-1+b_1-b_3}}{1-p^{b_2-b_3}} \\
&\quad + \frac{p^{-(1+a_2+b_3+2s)}}{1-p^{-(1+a_2+b_3+2s)}} \frac{1-p^{-1+a_1-a_3}}{1-p^{a_2-a_3}} \frac{1-p^{-1+b_1-b_2}}{1-p^{b_3-b_2}} \\
&\quad + \frac{p^{-(1+a_3+b_2+2s)}}{1-p^{-(1+a_3+b_2+2s)}} \frac{1-p^{-1+a_1-a_2}}{1-p^{a_3-a_2}} \frac{1-p^{-1+b_1-b_3}}{1-p^{b_2-b_3}} \\
&\quad + \frac{p^{-(1+a_3+b_3+2s)}}{1-p^{-(1+a_3+b_3+2s)}} \frac{1-p^{-1+a_1-a_2}}{1-p^{a_3-a_2}} \frac{1-p^{-1+b_1-b_2}}{1-p^{b_3-b_2}} \Big) (1-p^{a_1+b_1+2s-1}) \\
&\quad \times \Big(1 - \frac{1}{p^{1+a_2+b_2+2s)}}\Big)\Big(1 - \frac{1}{p^{1+a_2+b_3+2s)}}\Big)\Big(1 - \frac{1}{p^{1+a_3+b_2+2s)}}\Big)\Big(1 - \frac{1}{p^{1+a_3+b_3+2s)}}\Big).
\end{aligned} \tag{7.16}
$$

and thus $\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(s) = \prod_p \mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(p;s)$ where $\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(p;s)$ is defined by (6.3).

$\square$

Proof of Proposition 6.2. (i) In this proof we set

$$x_i = p^{-a_i}, \text{ for } 1 \le i \le 3, \tag{7.17}$$

$$y_j = p^{-b_j} \text{ for } 1 \le j \le 3, \text{ and} \tag{7.18}$$

$$u = p^{-1}. \tag{7.19}$$

We aim to show $\mathcal{A}_{\mathcal{I}_{\{a_1\}},\mathcal{J}_{\{b_1\}}}(0) = \mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(0)$. Note that $(\mathcal{I}_{\{a_1\}},\mathcal{J}_{\{b_1\}}) = (-b_1,a_2,a_3;-a_1,b_2,b_3)$. We observe that $\mathcal{A}_{\mathcal{I}_{\{a_1\}},\mathcal{J}_{\{b_1\}}}(0)$ is obtained from $\mathcal{A}_{\mathcal{I},\mathcal{J}}(0)$ by the transformation $a_1 \to -b_1$ and $b_1 \to -a_1$. Therefore

$$\mathcal{A}_{\mathcal{I}_{\{a_1\}},\mathcal{J}_{\{b_1\}}}(0) = \prod_p P(p^{b_1},p^{-a_2},p^{-a_3},p^{a_1},p^{-b_2},p^{-b_3},p^{-1}) = \prod_p P(y_1^{-1},x_2,x_3,x_1^{-1},y_2,y_3,u).$$

On the other hand,

$$C_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(0) = \prod_p Q(p^{-a_2},p^{-a_3},p^{-b_2},p^{-b_3};p^{-a_1},p^{-b_1};p^{-1},1) = \prod_p Q(x_2,x_3,y_2,y_3;x_1,y_1;u,1).$$

Thus (2.14) follows if $P(y_1^{-1},x_2,x_3,x_1^{-1},y_2,y_3,u) = Q(x_2,x_3,y_2,y_3;x_1,y_1;u,1)$. From the definitions (2.20) and (6.4) this identity would read

$$
\begin{aligned}
&1 - y_1^{-1}x_2x_3x_1^{-1}y_2y_3(y_1+x_2^{-1}+x_3^{-1})(x_1+y_2^{-1}+y_3^{-1})u^2\\
&+ y_1^{-1}x_2x_3x_1^{-1}y_2y_3\Big((y_1+x_2^{-1}+x_3^{-1})(y_1^{-1}+x_2+x_3)+(x_1+y_2^{-1}+y_3^{-1})(x_1^{-1}+y_2+y_3)-2\Big)u^3\\
&- y_1^{-1}x_2x_3x_1^{-1}y_2y_3(y_1^{-1}+x_2+x_3)(x_1^{-1}+y_2+y_3)u^4\\
&+ (y_1^{-1}x_2x_3x_1^{-1}y_2y_3)^2u^6\\
&= \Big(1+\Big(\frac{ux_2y_2}{1-ux_2y_2}\frac{1-ux_3x_1^{-1}}{1-x_3x_2^{-1}}\frac{1-uy_3y_1^{-1}}{1-y_3y_2^{-1}}\\
&+ \frac{ux_2y_3}{1-ux_2y_3}\frac{1-ux_3x_1^{-1}}{1-x_3x_2^{-1}}\frac{1-uy_2y_1^{-1}}{1-y_2y_3^{-1}}\\
&+ \frac{ux_3y_2}{1-ux_3y_2}\frac{1-ux_2x_1^{-1}}{1-x_2x_3^{-1}}\frac{1-uy_3y_1^{-1}}{1-y_3y_2^{-1}}\\
&+ \frac{ux_3y_3}{1-ux_3y_3}\frac{1-ux_2x_1^{-1}}{1-x_2x_3^{-1}}\frac{1-uy_2y_1^{-1}}{1-y_2y_3^{-1}}\Big)(1-ux_1^{-1}y_1^{-1})\Big)\\
&\times(1-ux_2y_2)(1-ux_2y_3)(1-ux_3y_2)(1-ux_3y_3).
\end{aligned}
$$

However, this may be verified by a Maple calculuation. [3]

(ii) We now establish $\mathcal{A}_{\mathcal{I},\mathcal{J}}(-a_1-b_1) = \mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\frac{a_1+b_1}{2})$. As above we have

$$
\mathcal{A}_{\mathcal{I},\mathcal{J}}(-a_1-b_1) = \prod_p P(x_1,x_2,x_3,y_1,y_2,y_3,ux_1^{-1}y^{-1}) \text{ and}
$$
$$
\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\tfrac{a_1+b_1}{2}) = \prod_p Q(x_2,x_3,y_2,y_3;x_1,y_1;u,x_1^{-1}y_1^{-1}).
$$

It suffices to verify $P(x_1,x_2,x_3,y_1,y_2,y_3,ux_1^{-1}y^{-1}) = Q(x_2,x_3,y_2,y_3;x_1,y_1;u,x_1^{-1}y_1^{-1})$. This too, was verified by a Maple calculation.

(iii) Lastly, we show $\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\frac{a_2+b_2}{2}) = \mathcal{C}_{-\mathcal{J},-\mathcal{I};\{-b_3\},\{-a_3\}}(\frac{b_2+a_2}{2})$. Now

$$
\mathcal{C}_{\mathcal{I},\mathcal{J};\{a_1\},\{b_1\}}(-\tfrac{a_2+b_2}{2}) = \prod_p Q(p^{-a_2},p^{-a_3},p^{-b_2},p^{-b_3};p^{-a_1},p^{-b_1};p^{-1},p^{a_2+b_2})
$$
$$
\mathcal{C}_{-\mathcal{J},-\mathcal{I};\{-b_3\},\{-a_3\}}(\tfrac{b_2+a_2}{2}) = \prod_p Q(p^{a_1},p^{a_2},p^{b_1},p^{b_2};p^{a_3},p^{b_3};p^{-1},p^{-a_2-b_2}).
$$

It suffices to verify $Q(x_2,x_3,y_2,y_3;x_1,y_1;u,x_2^{-1}y_2^{-1}) = Q(x_1^{-1},x_2^{-1},y_1^{-1},y_2^{-1};x_3^{-1},y_3^{-1},u,x_2y_2)$. Again this was checked with Maple. □

8. Appendix 2: Proofs of technical lemmas

In this section we prove several technical lemmas. We begin with a lemma which makes use of Stirling's formula.

8.1. **Proof of Lemma 2.2(ii).**

Proof. This argument follows closely [24, pp.390-391] We have

$$
\log\Gamma(z) = (z-\tfrac{1}{2})\log z - z + \frac{1}{2}\log(2\pi) + O(|z|^{-1}) \tag{8.1}
$$

[3] The Maple file is available upon request from the author.

where the branch of logarithm having argument in $(-\pi,\pi)$. Throughout this argument we use the notation

$$\alpha=\frac{1}{2}\Big(\frac{1}{2}+a+it\Big),\ \beta=\frac{s}{2},\ |a|\ll(\log T)^{-1}. \tag{8.2}$$

We begin by assuming $|\Im(s)|\le t^{\frac12}$. Note that

$$\log\Gamma(\alpha+\beta)-\log\Gamma(\alpha)=\beta\log(\alpha)+(\alpha+\beta-\tfrac12)\log(1+\beta/\alpha)-\beta+O(t^{-1}).$$

Also

$$(\alpha+\beta-\tfrac12)\log(1+\beta/\alpha)-\beta=(\alpha+\beta-\tfrac12)\Big(\frac{\beta}{\alpha}+O\Big(\Big(\frac{\beta}{\alpha}\Big)^2\Big)\Big)-\beta=O\Big(\frac{\beta^2}{\alpha}\Big)$$

and thus $\log\Gamma(\alpha+\beta)-\log\Gamma(\alpha)=O(\frac{|s|^2}{t})$. It follows that

$$\log\Gamma\Big(\frac12\Big(\frac12+a+it+s\Big)\Big)-\log\Gamma\Big(\frac12\Big(\frac12+a+it\Big)\Big)=\frac{s}{2}\log(\tfrac{it}{2})+O(\frac{|s|^2}{t}). \tag{8.3}$$

Conjugating the above equation and replacing a by $\overline{a}$ yields

$$\log\Gamma\Big(\frac12\Big(\frac12+a-it+s\Big)\Big)-\log\Gamma\Big(\frac12\Big(\frac12+a-it\Big)\Big)=\frac{s}{2}\log(-\tfrac{it}{2})+O(\frac{|s|^2}{t}). \tag{8.4}$$

Taking $a=a_j$ in (8.3) and $a=b_j$ in (8.4) we find

(8.5)

$$\log\Gamma\Big(\frac12\Big(\frac12+a_j+it+s\Big)\Big)-\log\Gamma\Big(\frac12\Big(\frac12+a_j+it\Big)\Big)+\log\Gamma\Big(\frac12\Big(\frac12+b_j-it+s\Big)\Big)-\log\Gamma\Big(\frac12\Big(\frac12+b_j-it\Big)\Big)$$
$$=\frac{s}{2}\log\Big(\Big(\frac{t}{2}\Big)^2\Big)+O(\frac{|s|^2}{t}).$$

Exponentiating and taking the product over $j=1,2,3$ yields

$$g_{\mathfrak{I},\mathfrak{J}}(s,t)=\prod_{j=1}^{3}\Big(\frac{t}{2}\Big)^s\Big(1+O\Big(\frac{|s|^2}{t}\Big)\Big)=\Big(\frac{t}{2}\Big)^{3s}\Big(1+O\Big(\frac{|s|^2}{t}\Big)\Big).$$

Next we deal with the case $|\Im(s)|>t^{\frac12}$. For convenience, we set $s=\sigma+iy$. We shall use repeatedly the Stirling estimate: for $0\le x\le 1$ and $|y|\ge 1$,

$$|\Gamma(x+iy)|=(2\pi)^{\frac12}|y|^{x-\frac12}e^{-\frac{\pi|y|}{2}}(1+O(|z|^{-1})). \tag{8.6}$$

Thus if $|y+t|\ge 1$, then

$$\left|\frac{\Gamma(\frac12(\frac12+a+it+s))}{\Gamma(\frac12(\frac12+a+it))}\right|\asymp\frac{|\frac{t+a''+y}{2}|^{\frac{a'+\sigma}{2}-\frac14}e^{-\frac{\pi}{4}|t+y+a''|}}{|\frac{t+a''}{2}|^{-\frac14}e^{-\frac{\pi(t+a'')}{4}}}\asymp\frac{|t+a''+y|^{\frac{a'+\sigma}{2}-\frac14}e^{-\frac{\pi}{4}|t+y+a''|}}{t^{-\frac14}e^{-\frac{\pi t}{4}}}$$

where $a=a'+ia''$. Similarly, if $|y-t|\ge 1$, then

$$\left|\frac{\Gamma(\frac12(\frac12+b-it+s))}{\Gamma(\frac12(\frac12+b-it))}\right|\asymp\frac{|b''+y-t|^{\frac{b'+\sigma}{2}-\frac14}e^{-\frac{\pi}{4}|b''+y-t|}}{|t|^{-\frac14}e^{-\frac{\pi t}{4}}} \tag{8.7}$$

Thus if $|y-t| \geq 1$ and $|y+t| \geq 1$, these combine to give

(8.8)
$$\left|\frac{\Gamma(\frac{1}{2}(\frac{1}{2}+a+it+s))\Gamma(\frac{1}{2}\left(\frac{1}{2}+b-it+s)\right)}{\Gamma(\frac{1}{2}(\frac{1}{2}+a+it))\Gamma(\frac{1}{2}(\frac{1}{2}+b-it))}\right| \ll \frac{|t+a''+y|^{\frac{a'+\sigma}{2}-\frac{1}{4}}|b''+y-t|^{\frac{b'+\sigma}{2}-\frac{1}{4}}e^{-\frac{\pi}{4}(|t+y+a''|+|b''+y-t|)}}{(|t|^{-\frac{1}{4}}e^{-\frac{\pi t}{4}})^2}$$
$$\ll \frac{|t+y|^{\frac{a'+\sigma}{2}-\frac{1}{4}}|y-t|^{\frac{b'+\sigma}{2}-\frac{1}{4}}e^{-\frac{\pi}{4}(|t+y|+|y-t|)}}{|t|^{-\frac{1}{2}}e^{-\frac{\pi t}{2}}}.$$

In the case that $|y-t| \geq 1$ and $|y+t| \geq 1$, we apply this with $a = a_j = a_j' + ia_j''$ and $b = b_j = b_j' + ib_j''$ to find that

$$|g_{\mathcal{I},\mathcal{J}}(s,t)| \ll \prod_{j=1}^{3} \frac{|y+t|^{\frac{a_j'+\sigma}{2}-\frac{1}{4}}|y-t|^{\frac{b_j'+\sigma}{2}-\frac{1}{4}}e^{-\frac{\pi}{4}(|t+y|+|y-t|)}}{t^{-\frac{1}{2}}e^{-\frac{\pi t}{2}}} \tag{8.9}$$

$$\ll \prod_{j=1}^{3} \frac{|y+t|^{\frac{\sigma}{2}-\frac{1}{4}}|y-t|^{\frac{\sigma}{2}-\frac{1}{4}}e^{-\frac{\pi}{4}(|t+y|+|y-t|)}}{t^{-\frac{1}{2}}e^{-\frac{\pi t}{2}}} \text{ if } y \ll t^{O(1)}. \tag{8.10}$$

If $y \geq t + (\log t)^2$, it follows from (8.9) and the decay of $e^{-\pi y}$ that this expression is $|g_{\mathcal{I},\mathcal{J}}(s,t)| \ll y^2 t^{3\sigma-1}$, uniformly for $\sigma \in [0,1]$. In the case $y \in [t+1, t+(\log t)^2]$ we have

$$\begin{aligned}|g_{\mathcal{I},\mathcal{J}}(s,t)| &\ll \Big(\frac{(y+t)^{\frac{\sigma}{2}-\frac{1}{4}}(y-t)^{\frac{\sigma}{2}-\frac{1}{4}}e^{-\frac{\pi y}{2}}}{t^{-\frac{1}{2}}e^{-\frac{\pi t}{2}}}\Big)^3 \ll t^{\frac{3}{4}+\frac{3\sigma}{2}}(y-t)^{\frac{3\sigma}{2}-\frac{3}{4}}e^{-\frac{3\pi}{2}(y-t)} \\ &\ll t^{\frac{3}{4}+\frac{3\sigma}{2}}(\log^2 t)^{\frac{3}{4}} \text{ since } \sigma \leq 1 \\ &\ll y^2 t^{3\sigma-1},\end{aligned}$$

since $y \geq t+1$ and $\sigma \in [0,1]$. For $y \in [\sqrt{t}, t-1]$ and $\sigma \in [\frac{1}{2}, 1]$, we have

$$|g_{\mathcal{I},\mathcal{J}}(s,t)| \ll \Big(\frac{(y+t)^{\frac{\sigma}{2}-\frac{1}{4}}(t-y)^{\frac{\sigma}{2}-\frac{1}{4}}e^{-\frac{\pi t}{2}}}{t^{-\frac{1}{2}}e^{-\frac{\pi t}{2}}}\Big)^3 \ll \Big(\frac{t^{\sigma-\frac{1}{2}}}{t^{-\frac{1}{2}}}\Big)^3 = t^{3\sigma} \ll y^2 t^{3\sigma-1},$$

since $y > \sqrt{t}$. For $y \in [\sqrt{t}, t-1]$ and $\sigma \in [0, \frac{1}{2})$, we obtain

$$|g_{\mathcal{I},\mathcal{J}}(s,t)| \ll \Big(\frac{(y+t)^{\frac{\sigma}{2}-\frac{1}{4}}(t-y)^{\frac{\sigma}{2}-\frac{1}{4}}e^{-\frac{\pi t}{2}}}{t^{-\frac{1}{2}}e^{-\frac{\pi t}{2}}}\Big)^3 \ll \Big(\frac{t^{\frac{\sigma}{2}-\frac{1}{4}}(t-y)^{\frac{\sigma}{2}-\frac{1}{4}}}{t^{-\frac{1}{2}}}\Big)^3$$
$$\ll t^{\frac{3\sigma}{2}+\frac{3}{4}}(t-y)^{\frac{3\sigma}{2}-\frac{3}{4}} \ll y^2 t^{3\sigma-1}. \tag{8.11}$$

This is straightforward in the case $\sigma = \frac{1}{2}$ since $t \ll y^2$. In the case that $\sigma \in [0, \frac{1}{2})$ this can be checked by considering the function $h(y) = y^2(t-y)^{\frac{3}{4}-\frac{3\sigma}{2}}$ on the interval $[\sqrt{t}, t-1]$. Elementary calculus shows that the minimum of h on this interval is $\gg t^{\frac{7}{4}-\frac{3\sigma}{2}}$ and therefore (8.11) follows. Now if $y \in [t-1, t+1]$, then $|\frac{\Gamma(\frac{1}{2}(\frac{1}{2}+b-it+s))}{\Gamma(\frac{1}{2}(\frac{1}{2}+b-it))}| \ll 1$ and thus

$$\left|\frac{\Gamma(\frac{1}{2}(\frac{1}{2}+a+it+s))\Gamma(\frac{1}{2}\left(\frac{1}{2}+b-it+s)\right)}{\Gamma(\frac{1}{2}(\frac{1}{2}+a+it))\Gamma(\frac{1}{2}(\frac{1}{2}+b-it))}\right| \ll \frac{|t+a'+y|^{\frac{\sigma}{2}-\frac{1}{4}}e^{-\frac{\pi}{2}|t+y+a''|}}{|t|^{-\frac{1}{4}}e^{-\frac{\pi t}{4}}} \ll t^{\frac{\sigma}{2}}e^{-\frac{\pi y}{2}}. \tag{8.12}$$

Therefore $|g_{\mathcal{I},\mathcal{J}}(s,t)| \ll (t^{\frac{\sigma}{2}}e^{-\frac{\pi y}{2}})^3 \ll y^2 t^{3\sigma-1}$, since $y \in [t-1, t+1]$. The cases for $y \leq -\sqrt{t}$ are proven in a similar fashion. □

8.2. **Partial derivative bounds.** We now provide the bound for the partial derivatives of $f(x,y)$, defined in (5.9), which occurs in the proof of Proposition 5.1.

Lemma 8.1. *For $\Re(s) = \varepsilon$, $|\Im(s)| \le \sqrt{T}$, $i \ge 0$,*

$$\frac{d^i}{dt^i} g_{\mathfrak{I},\mathfrak{J}}(s,t) \ll_i |s|^i T^{3\varepsilon - i} \tag{8.13}$$

Proof. Let $p_j(s,t) = \frac{\Gamma(\frac{1}{2}(\frac{1}{2}+a_j+it+s))\Gamma(\frac{1}{2}(\frac{1}{2}+b_j-it+s))}{\Gamma(\frac{1}{2}(\frac{1}{2}+a_j+it))\Gamma(\frac{1}{2}(\frac{1}{2}+b_j-it))}$, $\theta_j(s,t) = \frac{d}{dt}\log p_j(s,t)$, and $\Theta(s,t) = \sum_{j=1}^{3} \theta_j(s,t)$. Observe that

$$\frac{d}{dt} g_{\mathfrak{I},\mathfrak{J}}(s,t) = g_{\mathfrak{I},\mathfrak{J}}(s,t)\Theta(s,t) \tag{8.14}$$

and more generally, for $i \ge 1$,

$$\frac{d^i}{dt^i} g_{\mathfrak{I},\mathfrak{J}}(s,t) = \sum_{u+v=i-1} \binom{i-1}{u} \frac{d^u}{dt^u} g_{\mathfrak{I},\mathfrak{J}}(s,t) \frac{d^v}{dt^v}\Theta(s,t). \tag{8.15}$$

We now demonstrate that

$$\frac{d^v}{dt^v}\Theta(s,t) \ll |s| t^{-v-1} \tag{8.16}$$

which follows from

$$\frac{d^v}{dt^v}\theta_j(s,t) \ll |s| t^{-v-1} \text{ for } j = 1,2,3. \tag{8.17}$$

Using these facts, we can prove the Lemma by induction. Observe that Lemma 2.2 (ii), (8.16) with $v = 0$, and (8.14) imply $\frac{d}{dt} g(s,t) \ll |s| T^{3\varepsilon - 1}$. This establishes the Lemma in the case $i = 1$. Now assume the inductive hypothesis, $\frac{d^u}{dt^u} g(s,t) \ll |s|^u T^{3\varepsilon - u}$ for $u \le i-1$. Combining this with (8.16) and (8.15), we obtain (8.13) for all $i \ge 0$, $\Re(s) = \varepsilon$, and $|\Im(s)| \le \sqrt{T}$.

Thus to complete the proof we must establish (8.17). Note that

$$\theta_j(s,t) = \frac{i}{2}\Bigg(\frac{\Gamma'(\frac{1}{2}(\frac{1}{2}+a_j+it+s))}{\Gamma(\frac{1}{2}(\frac{1}{2}+a_j+it+s))} - \frac{\Gamma'(\frac{1}{2}(\frac{1}{2}+a_j+it))}{\Gamma(\frac{1}{2}(\frac{1}{2}+a_j+it))} + \frac{\Gamma'(\frac{1}{2}(\frac{1}{2}+b_j-it+s))}{\Gamma(\frac{1}{2}(\frac{1}{2}+b_j-it+s))} - \frac{\Gamma'(\frac{1}{2}(\frac{1}{2}+b_j-it))}{\Gamma(\frac{1}{2}(\frac{1}{2}+b_j-it))} \Bigg).$$

However, we have the asymptotic expansion $\frac{\Gamma'}{\Gamma}(z) = \log z + O(|z|^{-1})$ and thus

$$\begin{aligned}
\theta_j(s,t) &= \frac{i}{2}\Big(\log\Big(\frac{1}{2}(\frac{1}{2}+a_j+it)\Big) - \log\Big(\frac{1}{2}(\frac{1}{2}+a_j+it+s)\Big) \\
&\quad + \log\Big(\frac{1}{2}(\frac{1}{2}+b_j-it+s)\Big) - \log\Big(\frac{1}{2}(\frac{1}{2}+b_j-it)\Big)\Big) + O(t^{-1}) \\
&= \frac{i}{2}(\log(t+y) - \log(t) + \log|y-t| - \log(t)) + O(t^{-1}) \\
&= \frac{i}{2}(\log\Big(1+\frac{y}{t}\Big) + \log\Big(1-\frac{y}{t}\Big)) + O(t^{-1}) \ll \frac{y}{t}
\end{aligned}$$

since $|y| \le \sqrt{T}$. We now study the higher derivatives of θ_j. We have

$$\begin{aligned}
\frac{d^v}{dt^v}\theta_j(s,t) = \Big(\frac{i}{2}\Big)^{v+1} \Big(\Big(\frac{\Gamma'}{\Gamma}\Big)^{(v)}(\frac{1}{2}(\frac{1}{2}+a_j+it+s)) - \Big(\frac{\Gamma'}{\Gamma}\Big)^{(v)}(\frac{1}{2}(\frac{1}{2}+a_j+it)) \\
+ \Big(\frac{\Gamma'}{\Gamma}\Big)^{(v)}(\frac{1}{2}(\frac{1}{2}+b_j-it+s)) - \Big(\frac{\Gamma'}{\Gamma}\Big)^{(v)}(\frac{1}{2}(\frac{1}{2}+b_j-it))\Big)
\end{aligned}$$

It is known that (see [1])

$$\Big(\frac{\Gamma'}{\Gamma}\Big)^{(v)}(z) = \frac{(-1)^{v-1}(v-1)!}{z^v} + O\Big(\frac{1}{|z|^{v+1}}\Big) \tag{8.18}$$

so that

$$\frac{d^v}{dt^v}\theta_j(s,t) = (-1)^{v-1}(v-1)!\Big(\frac{i}{2}\Big)^{v+1}\Big(\frac{1}{(\frac{1}{2}(\frac{1}{2}+a_j+it+s))^v} - \frac{1}{(\frac{1}{2}(\frac{1}{2}+a_j+it))^v} + \frac{1}{(\frac{1}{2}(\frac{1}{2}+b_j-it+s))^v} - \frac{1}{(\frac{1}{2}(\frac{1}{2}+b_j-it))^v}\Big) + O(t^{-v-1}).$$

Writing $s=\varepsilon+iy$, $a_j=a_j'+ia_j''$, $R=\frac{1}{2}(\frac{1}{2}+a_j'+\varepsilon)$, and let L is the straight line from $R+\frac{i}{2}(t+a_j'')$ to $R+\frac{i}{2}(t+y+a_j'')$ of length $|y|$. Thus

$$\Big|\frac{1}{(\frac{1}{2}(\frac{1}{2}+a_j+it+s))^v} - \frac{1}{(\frac{1}{2}(\frac{1}{2}+a_j+it))^v}\Big| = \Big|(\frac{1}{R+\frac{i}{2}(t+y+a_j''))^v} - \frac{1}{(R+\frac{i}{2}(t+a_j''))^v}\Big| = v\Big|\int_L z^{-v-1}dz\Big|$$
$$\le v|y|\max_{z\in L}|z|^{-v-1} \ll v|y|t^{-v-1},$$

since $|y|\le\sqrt{T}$. In a similar fashion, we can show that

$$\Big|\frac{1}{(\frac{1}{2}(\frac{1}{2}+b_j-it+s))^v} - \frac{1}{(\frac{1}{2}(\frac{1}{2}+b_j-it))^v}\Big| \ll v|y|t^{-v-1}.$$

Combining these facts we derive (8.17).

□

Lemma 8.2. *Let $M\asymp N$ and $(x,y)\in[M,2M]\times[N,2N]$. Then*

$$x^m y^n f^{(m,n)}(x,y) \ll P^{m+n} \text{ where } P=\Big(\frac{M}{rT_0}+\frac{T}{T_0}\Big)T^\varepsilon \tag{8.19}$$

where we recall that f is defined by (5.9).

Proof. Let $f(x,y)=W(\frac{x}{M})W(\frac{y}{N})\phi(x,y)$ where

$$\phi(x,y)=\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3xy}\Big)^s\frac{1}{T}\int_{-\infty}^{\infty}\Big(1+\frac{r}{y}\Big)^{-it}g_{\mathfrak{J},\mathfrak{J}}(s,t)w(t)dt\,ds$$

Observe that for $i\ge 0$

$$\frac{d^i}{dx^i}W\Big(\frac{x}{M}\Big) \ll W^{(i)}\Big(\frac{x}{M}\Big)M^{-i} \ll M^{-i} \text{ and } \frac{d^i}{dx^i}W\Big(\frac{y}{N}\Big) \ll W^{(i)}\Big(\frac{y}{N}\Big)N^{-i} \ll N^{-i} \ll M^{-i}. \tag{8.20}$$

We shall prove that

$$x^m y^n \phi^{(m,n)}(x,y) \ll P^{m+n} \text{ where } P=\Big(\frac{M}{rT_0}+\frac{T}{T_0}\Big)T^\varepsilon. \tag{8.21}$$

By the generalized product rule, applied twice,

$$\begin{aligned} f^{(m,n)}(x,y) &= \sum_{i_1+i_2=m}\binom{m}{i_1}W^{(i_1)}\Big(\frac{x}{M}\Big)M^{-i_1}\sum_{j_1+j_2=n}\binom{n}{j_1}W^{(j_1)}\Big(\frac{y}{N}\Big)N^{-j_1}\phi^{(i_2,j_2)}(x,y) \\ &\ll \sum_{i_1+i_2=m}\binom{m}{i_1}M^{-i_1}\sum_{j_1+j_2=n}\binom{n}{j_1}N^{-j_1}\Big(\frac{P}{x}\Big)^{i_2}\Big(\frac{P}{y}\Big)^{j_2} \\ &\ll M^{-m}N^{-n}\sum_{i_1+i_2=m}\binom{m}{i_1}P^{i_2}\sum_{j_1+j_2=n}\binom{n}{j_1}P^{j_2} \\ &= M^{-m}N^{-n}(1+P)^m(1+P)^n \end{aligned} \tag{8.22}$$

where we have used $x\asymp M$ and $y\asymp N$. Since $P\ge 1$, $M\asymp x$, and $N\asymp y$, we obtain (8.19). We now reduced the proof of the lemma to establishing (8.21). It will be convenient to compute the derivatives of x^{-s} and

y^{-s}. Define a sequence of polynomials $P_j(s)$ by $P_0(s)=1$ and $P_j(s)=\prod_{i=0}^{j-1}(s+i)$ for $j\geq 1$. Note that $\frac{d^k}{dx^k}x^{-s}=(-1)^kP_k(s)x^{-s-k}$. Observe that

$$\phi^{(m,n)}(x,y)=\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\Big(\frac{1}{\pi^3}\Big)^s\frac{1}{T}\int_{-\infty}^{\infty}\frac{d^m}{dx^m}\frac{d^n}{dy^n}\Big(x^{-s}y^{-s}\Big(1+\frac{r}{y}\Big)^{-it}\Big)g_{\mathfrak{I},\mathfrak{J}}(s,t)\omega(t)dt\,ds. \tag{8.23}$$

By the generalized product rule

$$\begin{aligned}\frac{d^m}{dx^m}\frac{d^n}{dy^n}\Big(x^{-s}y^{-s}\Big(1+\frac{r}{y}\Big)^{-it}\Big)&=\frac{d^m}{dx^m}(x^{-s})\frac{d^n}{dy^n}\Big(y^{-s}\Big(1+\frac{r}{y}\Big)^{-it}\Big)\\&=(-1)^mP_m(s)x^{-s-m}\sum_{\substack{u+v=n\\u,v\geq 0}}\binom{n}{u}(-1)^uP_u(s)y^{-s-u}\frac{d^v}{dy^v}\Big(1+\frac{r}{y}\Big)^{-it}.\end{aligned} \tag{8.24}$$

It suffices to determine $\frac{d^v}{dy^v}(1+\frac{r}{y})^{-it}$ where $0\leq v\leq n$. Write $(1+\frac{r}{y})^{-it}=F(G(y))$ where $F(y)=e^{-ity}$ and $G(y)=\log(1+\frac{r}{y})$. By the generalized chain rule (Faà di Bruno's formula)

$$\frac{d^v}{dy^v}\Big(1+\frac{r}{y}\Big)^{-it}=\frac{d^v}{dy^v}F(G(y))=\sum_{\substack{1\cdot m_1+\cdots+vm_v=v\\ \mathbf{m}=(m_1,\ldots,m_v)\in(\mathbb{Z}_{\geq 0})^v}}c_n(\mathbf{m})F^{(m_1+\cdots+m_v)}(G(y))\prod_{j=1}^{n}(G^{(j)}(y))^{m_j}, \tag{8.25}$$

where $c_n(\mathbf{m})=\frac{n!}{m_1!1!^{m_1}\cdots m_v!v!^{m_v}}$. We must now calculate the derivatives $F^{(j)}(y)$ and $G^{(j)}(y)$. We have $F^{(j)}(y)=(-it)^je^{-ity}$ and thus $F^{(m_1+\cdots+m_v)}(G(y))=(-it)^{m_1+\cdots+m_v}(1+\frac{r}{y})^{-it}$. Observe that $G(y)=\log(r+y)-\log(y)$. It follows that $G^{(j)}(y)=(-1)^{j-1}(j-1)!((y+r)^{-j}-y^{-j})$ and by the mean value theorem

$$G^{(j)}(y)\ll_j\frac{r}{y^{j+1}}. \tag{8.26}$$

Using the above facts

$$\frac{d^v}{dy^v}\Big(1+\frac{r}{y}\Big)^{-it}=\Big(1+\frac{r}{y}\Big)^{-it}\sum_{\substack{1\cdot m_1+\cdots+vm_v=v\\ \mathbf{m}=(m_1,\ldots,m_v)\in(\mathbb{Z}_{\geq 0})^v}}c_n(\mathbf{m})(-it)^{m_1+\cdots+m_v}\prod_{j=1}^{v}(G^{(j)}(y))^{m_j}. \tag{8.27}$$

Next note that

$$\prod_{j=1}^{v}(G^{(j)}(y))^{m_j}\asymp\prod_{j=1}^{v}\Big(\frac{r}{y^{j+1}}\Big)^{m_j}=\Big(\frac{r}{y}\Big)^{m_1+\cdots+m_v}\frac{1}{y^{1\cdot m_1+\cdots v\cdot m_v}}. \tag{8.28}$$

Furthermore, observe that $1\leq\sum_{j=1}^{v}m_j\leq v$. We group together those $\mathbf{m}$ such that $k=\sum_{j=1}^{v}m_j$ and obtain from (8.26) that

$$\prod_{j=1}^{v}(G^{(j)}(y))^{m_j}\asymp\Big(\frac{r}{y}\Big)^{k}y^{-v}. \tag{8.29}$$

Thus (8.27), (8.28), and (8.29) imply

$$\frac{d^v}{dy^v}\Big(1+\frac{r}{y}\Big)^{-it}=\Big(1+\frac{r}{y}\Big)^{-it}y^{-v}\sum_{k=1}^{v}t^kh_{k;v}(y) \tag{8.30}$$

where $h_{k;v}(y)$ are smooth functions on $[M,2M]$ satisfying

$$h_{k;v}(y)\asymp\Big(\frac{r}{y}\Big)^{k}. \tag{8.31}$$

Therefore

$$
\begin{aligned}
&\frac{d^m}{dx^m}\frac{d^n}{dy^n}\Big(x^{-s}y^{-s}\Big(1+\frac{r}{y}\Big)^{-it}\Big)\\
&=(-1)^m P_m(s)x^{-s-m}\sum_{\substack{u+v=n\\u,v\geq 0}}\binom{n}{u}(-1)^u P_u(s)y^{-s-u}\Big(\Big(1+\frac{r}{y}\Big)^{-it}y^{-v}\sum_{k=1}^{v}t^k h_{k;v}(y)\Big)\\
&=(-1)^m P_m(s)x^{-s-m}y^{-s-n}\Bigg(\sum_{\substack{u+v=n\\u,v\geq 0}}\binom{n}{u}(-1)^u P_u(s)\sum_{k=1}^{v}t^k h_{k;v}(y)\Bigg)\Big(1+\frac{r}{y}\Big)^{-it}.
\end{aligned}\tag{8.32}
$$

From (8.23) and the last identity, it follows that

$$
x^m y^n \phi^{(m,n)}(x,y)=(-1)^m\sum_{u+v=n}\binom{n}{u}\frac{(-1)^u}{2\pi i}\int_{(\varepsilon)}\frac{G(s)P_m(s)P_u(s)}{s}\Big(\frac{1}{\pi^3 xy}\Big)^s\mathscr{I}_v(s)\,ds\tag{8.33}
$$

where

$$
\mathscr{I}_v(s)=\frac{1}{T}\int_{-\infty}^{\infty}\Big(1+\frac{r}{y}\Big)^{-it}\mathbf{h}_v(t,y)g(s,t)\omega(t)dt,\tag{8.34}
$$

$$
\mathbf{h}_v(t,y)=\sum_{k=1}^{v}t^k h_{k;v}(y).\tag{8.35}
$$

We begin by bounding the portion of the integral in (8.33) with $|\Im(s)|\geq\sqrt{T}$. Thus we bound $\mathscr{I}_v(s)$, assuming $|\Im(s)|\geq\sqrt{T}$. We have

$$
\begin{aligned}
|\mathscr{I}_v(s)|&\ll\frac{1}{T}\int_{-\infty}^{\infty}\sum_{k=1}^{v}t^k\Big(\frac{r}{y}\Big)^k\Big(\frac{t}{2}\Big)^{3\varepsilon}(1+O(|s|^2t^{-1})\omega(t)dt\\
&\ll\Big(\frac{T}{2}\Big)^{3\varepsilon}\sum_{k=1}^{v}\Big(\frac{Tr}{y}\Big)^k\frac{1}{T}\int_{-\infty}^{\infty}(1+|s|^2t^{-1})\omega(t)dt\\
&\ll\Big(\frac{T}{2}\Big)^{3\varepsilon}\sum_{k=1}^{v}\Big(\frac{Tr}{y}\Big)^k|s|^2T^{-1},
\end{aligned}\tag{8.36}
$$

since $|\Im(s)|\geq\sqrt{T}$. Hence the portion of (8.33) with $|\Im(s)|\geq\sqrt{T}$ is

$$
\begin{aligned}
&\sum_{u+v=n}\binom{n}{u}\int_{\substack{\Re(s)=\varepsilon\\|\Im(s)|\geq\sqrt{T}}}\frac{|G(s)P_m(s)P_u(s)|}{|s|}\Big(\frac{1}{\pi^3xy}\Big)^{\varepsilon}\Big(\frac{T}{2}\Big)^{3\varepsilon}\sum_{k=1}^{v}\Big(\frac{Tr}{y}\Big)^k|s|^2T^{-1}\\
&\ll\Big(\frac{T}{2\pi^3xy}\Big)^{3\varepsilon}T^{-1}\sum_{u+v=n}\binom{n}{u}\sum_{k=1}^{v}\Big(\frac{Tr}{y}\Big)^k\int_{\substack{\Re(s)=\varepsilon\\|\Im(s)|\geq\sqrt{T}}}|s|^{-B+m+n+1}|ds|\\
&\ll\Big(\frac{T}{2\pi^3xy}\Big)^{3\varepsilon}T^{-1}(\sqrt{T})^{-B+m+n+2}\sum_{u+v=n}\binom{n}{u}\sum_{k=1}^{v}\Big(\frac{Tr}{y}\Big)^k\\
&\ll\Big(\frac{T}{2\pi^3xy}\Big)^{3\varepsilon}(\sqrt{T})^{-B+m+n}\sum_{u+v=n}\binom{n}{u}\Big(\frac{T}{T_0}T^{\varepsilon}\Big)^v,
\end{aligned}\tag{8.37}
$$

since $\frac{rT}{y}\ll\frac{rT}{M}\ll\frac{T}{T_0}T^{\varepsilon}$ as $r\ll\frac{M}{T_0}T^{\varepsilon}$. Therefore this is

$$
\Big(\frac{T}{2\pi^3xy}\Big)^{3\varepsilon}(\sqrt{T})^{-B+m+n}\Big(1+\frac{T}{T_0}T^{\varepsilon}\Big)^n\ll T^{3\varepsilon}T^{-m-n}\Big(\frac{T}{T_0}T^{\varepsilon}\Big)^n\ll T^{3\varepsilon n}T^{-m}T_0^{-n}.
$$

by the choice $B = 3(m+n)$. Thus we have

(8.38)
$$x^m y^n \phi^{(m,n)}(x,y) = (-1)^m \sum_{u+v=n} \binom{n}{u} \frac{(-1)^u}{2\pi i} \int_{\substack{\Re(s)=\varepsilon \\ |\Im(s)|\le\sqrt{T}}} \frac{G(s)P_m(s)P_u(s)}{s} \Big(\frac{1}{\pi^3 xy}\Big)^s \mathscr{I}_v + O(T^{\varepsilon(3n)} T^{-m} T_0^{-n}).$$

We now provide a bound for $\mathscr{I}_v(s)$ in the case $|\Im(s)| \le \sqrt{T}$. For any $\ell \ge 0$, integration by parts implies

$$\mathscr{I}_v(s) = \frac{(-1)^\ell}{T} \int_{-\infty}^{\infty} \frac{\Big(1+\frac{r}{y}\Big)^{-it}}{(-i\log(1+\frac{r}{y}))^\ell} \frac{d^\ell}{dt^\ell}\Big(\mathbf{h}_v(t,y) g(s,t)\omega(t)\Big) dt.$$

Setting $\ell = m+n$ we have

$$\mathscr{I}_v(s) = \frac{(-1)^{m+n}}{T} \int_{-\infty}^{\infty} \frac{\Big(1+\frac{r}{y}\Big)^{-it}}{(-i\log(1+\frac{r}{y}))^{m+n}} \frac{d^{m+n}}{dt^{m+n}}\Big(\mathbf{h}_v(t,y) g(s,t)\omega(t)\Big) dt \tag{8.39}$$

and by taking absolute values

$$|\mathscr{I}_v(s)| \ll \Big(\frac{y}{r}\Big)^{m+n} \frac{1}{T} \int_{T_1}^{T_2} \Big| \frac{d^{m+n}}{dt^{m+n}}\Big(\mathbf{h}_v(t,y) g(s,t)\omega(t)\Big)\Big| dt, \tag{8.40}$$

since $\log(1+x) \asymp |x|$ for $|x| \le \frac{1}{2}$ and $\frac{r}{y} \ll \frac{T^\varepsilon}{T_0} \le \frac{1}{2}$. By the generalized product rule

$$\frac{d^{m+n}}{dt^{m+n}}\Big(\mathbf{h}_v(t,y) g(s,t)\omega(t)\Big) = \sum_{\substack{i_1+i_2+i_3=m+n \\ \mathbf{i}=(i_1,i_2,i_3)\in(\mathbb{Z}_{\ge 0})^3}} \binom{m+n}{i_1,i_2,i_3} \frac{d^{i_1}}{dt^{i_1}}\mathbf{h}_v(t,y) \frac{d^{i_2}}{dt^{i_2}} g(s,t) \frac{d^{i_3}}{dt^{i_3}}\omega(t).$$

By Lemma 8.1 and $\frac{d^{i_3}}{dt^{i_3}}\omega(t) \ll T_0^{-i_3}$ it follows that

$$\begin{aligned}\frac{d^{m+n}}{dt^{m+n}}\Big(\mathbf{h}_v(t,y) g(s,t)\omega(t)\Big) &\ll \sum_{\substack{i_1+i_2+i_3=m+n \\ i_1\le v}} \binom{m+n}{i_1,i_2,i_3} \Big(\sum_{k=i_1}^{v} t^{k-i_1}|h_{k;v}(y)|\Big) |s|^{i_2} t^{3\varepsilon-i_2} T_0^{-i_3} \\ &\ll T^{3\varepsilon} \sum_{\substack{i_1+i_2+i_3=m+n \\ i_1\le v}} \Big(\sum_{k=i_1}^{v} T^{k-i_1}\Big(\frac{r}{y}\Big)^k\Big) T^{-i_2} T_0^{-i_3}\end{aligned}$$

by (8.31) and since $\frac{d^{i_1}}{dt^{i_1}}\mathbf{h}_v(t,y) = 0$ for $i_1 > v$. Inserting this is (8.40),

$$\mathscr{I}_v \ll |s|^{m+n} T^{3\varepsilon} \sum_{\substack{i_1+i_2+i_3=m+n \\ i_1\le v}} \Big(\sum_{k=i_1}^{v} T^{k-i_1}\Big(\frac{y}{r}\Big)^{m+n-k}\Big) T^{-i_2} T_0^{-i_3}.$$

Therefore

$$\begin{aligned} &x^m y^n \phi^{(m,n)}(x,y) \\ &\ll \Big(\frac{T^3}{xy}\Big)^\varepsilon \sum_{u+v=n} \binom{n}{u} \int_{(\varepsilon)} \Big|\frac{G(s)P_m(s)P_u(s)}{s}\Big| \sum_{\substack{i_1+i_2+i_3=m+n \\ i_1\le v}} \Big(\sum_{k=i_1}^{v} T^{k-i_1}\Big(\frac{y}{r}\Big)^{m+n-k}\Big) T^{-i_2} T_0^{-i_3} d|s| \\ &\ll \Big(\frac{T^3}{xy}\Big)^\varepsilon \sum_{u+v=n} \binom{n}{u} \sum_{\substack{i_1+i_2+i_3=m+n \\ i_1\le v}} \Big(\sum_{k=i_1}^{v} T^{k-i_1}\Big(\frac{y}{r}\Big)^{m+n-k}\Big) T^{-i_2} T_0^{-i_3}. \end{aligned} \tag{8.41}$$

Now

$$
\begin{aligned}
&\sum_{\substack{i_1+i_2+i_3=m+n\\ i_1\le v}} \Big(\sum_{k=i_1}^{v} T^{k-i_1}\Big(\frac{y}{r}\Big)^{m+n-k}\Big) T^{-i_2}T_0^{-i_3}\\
&= \sum_{i_1\le v}\Big(\sum_{k=i_1}^{v} T^{k-i_1}\Big(\frac{y}{r}\Big)^{m+n-k}\Big)\Big(\sum_{i_2+i_3=m+n-i_1} T^{-i_2}T_0^{-i_3}\Big)\\
&\ll \sum_{i_1\le v}\Big(\sum_{k=i_1}^{v} T^{k-i_1}\Big(\frac{y}{r}\Big)^{m+n-k}\Big)\Big(\frac{1}{T}+\frac{1}{T_0}\Big)^{m+n-i_1}\\
&= \sum_{k=0}^{v}\Big(\frac{y}{r}\Big)^{m+n-k}\sum_{i_1=0}^{k} T^{k-i_1}\Big(\frac{1}{T}+\frac{1}{T_0}\Big)^{m+n-i_1}\\
&\ll \sum_{k=0}^{v}\Big(\frac{y}{r}\Big)^{m+n-k}\sum_{i_1=0}^{k} T^{k-i_1}\Big(\frac{1}{T_0}\Big)^{m+n-i_1}\\
&\ll \sum_{k=0}^{v}\Big(\frac{y}{r}\Big)^{m+n-k} T^{k}\Big(\frac{1}{T_0}\Big)^{m+n}.
\end{aligned}
$$

Inserting this in (8.41)

$$
\begin{aligned}
x^m y^n \phi^{(m,n)}(x,y) &\ll \Big(\frac{T^3}{xy}\Big)^{\varepsilon}\frac{1}{T_0^{m+n}}\sum_{u+v=n}\binom{n}{u}\sum_{k=0}^{v}\Big(\frac{y}{r}\Big)^{m+n-k}T^k\\
&= \Big(\frac{T^3}{xy}\Big)^{\varepsilon}\frac{1}{T_0^{m+n}}\sum_{u+v=n}\binom{n}{u}\Big(\frac{y}{r}\Big)^{m+n-v}\sum_{k=0}^{v}\Big(\frac{y}{r}\Big)^{v-k}T^k\\
&\ll \Big(\frac{T^3}{xy}\Big)^{\varepsilon}\frac{1}{T_0^{m+n}}\sum_{u+v=n}\binom{n}{u}\Big(\frac{y}{r}\Big)^{m+n-v}\Big(\frac{y}{r}+T\Big)^{v}\\
&= \Big(\frac{T^3}{xy}\Big)^{\varepsilon}\frac{1}{T_0^{m+n}}\Big(\frac{y}{r}\Big)^{m}\Big(2\frac{y}{r}+T\Big)^{n}\\
&\ll T^{3\varepsilon}\Big(\frac{y}{r}+T\Big)^{m+n}T_0^{-m-n},
\end{aligned}
$$

as desired. $\square$

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