Graphs/relations in $\mathbb{R}^2$ that should be familiar at the beginning of your University career.

1. Lines
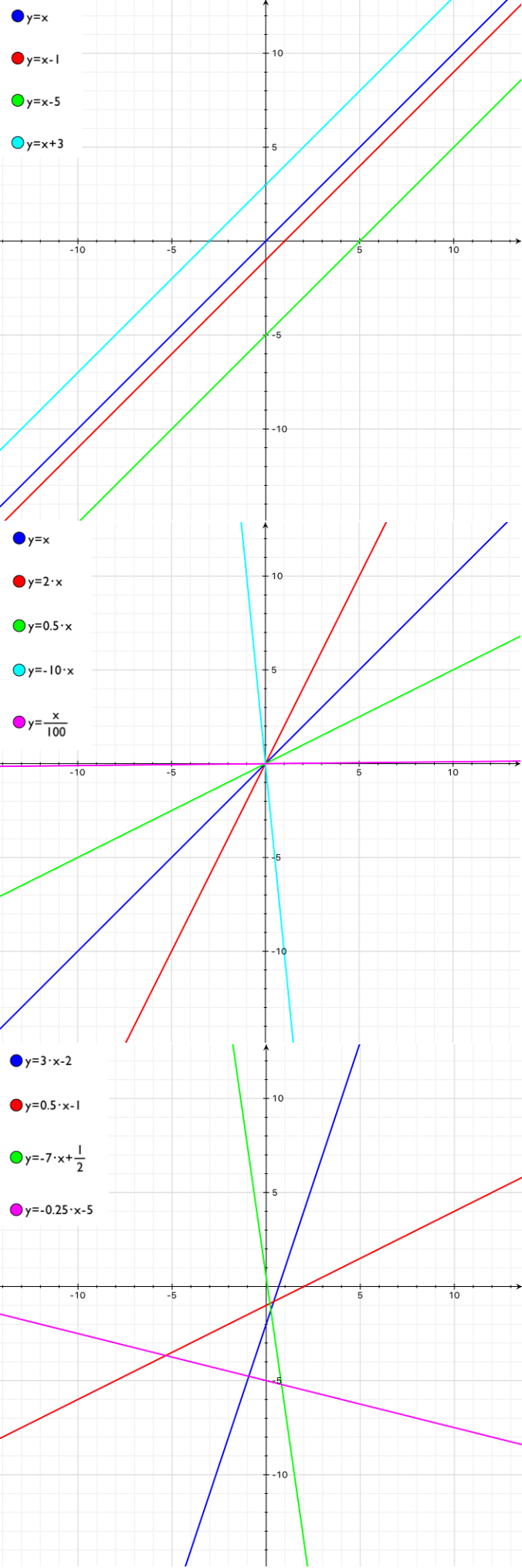
Slope/intercept formula: $y = mx + b$, where $m$ is the slope and $b$ is the $y-$interept.

Point/slope formula: $y - y_0 = m(x - x_0)$, where $(x_0, y_0)$ is a point on the line and $m$ is the slope

Slope formula: $m = \frac{y_1 - y_0}{x_1 - x_0}$, where $(x_0, y_0)$ and $(x_1, y_1)$ are points on the line

The domain and range of lines is always $\mathbb{R}$

Graphed Examples:



- y=x
- y=x-1
- y=x-5
- y=x+3



- y=x
- y=2·x
- y=0.5·x
- y=-10·x
- $y=\frac{x}{100}$



- y=3·x-2
- y=0.5·x-1
- $y=-7\cdot x+\frac{1}{2}$
- y=-0.25·x-5

2. Polynomials

Any equation of the form $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

the domain of all polynomials is $\mathbb{R}$.

a. Degree 2 polynomials/Parabolas/Quadratics

Standard form: $\qquad y = ax^2 + bx + c$,

where $a$ and $b$ are the coefficients of $x^2$ and $x$ respectively

$c$ is a constant (or I suppose we could call it the coefficient of $x^0$)

$ax^2, bx$, and $c$ are terms, in particular they are the degree 2, degree 1, and degree 0 terms

$x = -\dfrac{b}{2a}$ is the axis of symmetry

Vertex form: $\qquad y = a(x - h)^2 + k$,

where $(h, k)$ is the vertex of the parabola

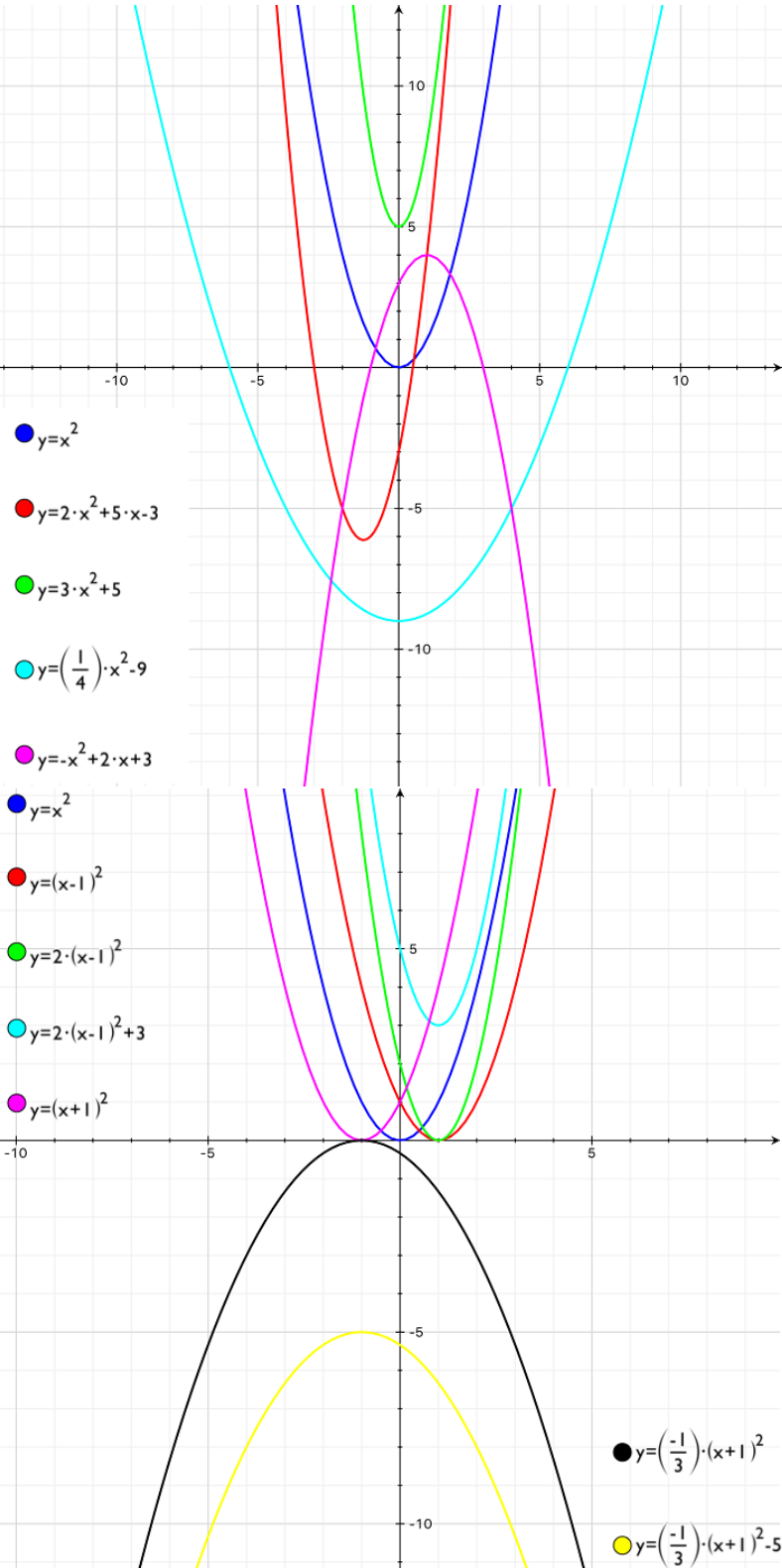if $a < 0$ then the parabola opens downwards

if $a = 0$ then this isn't a parabola it is the vertical line $y = k$

if $a > 0$ then the parabola opens upwards

naturally $x = h$ is this axis of symmetry

The range of a parabola depends on whether the vertex is an absolute minimum or an absolute maximum. If it is an absolute minimum then the range is the interval $[k, \infty)$. If it is an absolute maximum then the range is the interval $(-\infty, k]$.

Graphed Examples:



$\bullet \ y=x^2$

$\bullet \ y=2 \cdot x^2+5 \cdot x-3$

$\bullet \ y=3 \cdot x^2+5$

$\bigcirc \ y=\left(\dfrac{1}{4}\right) \cdot x^2-9$

$\bullet \ y=-x^2+2 \cdot x+3$

$\bullet \ y=x^2$

$\bullet \ y=(x-1)^2$

$\bigcirc \ y=2 \cdot (x-1)^2$

$\bigcirc \ y=2 \cdot (x-1)^2+3$

$\bullet \ y=(x+1)^2$

$\bullet \ y=\left(\dfrac{-1}{3}\right) \cdot (x+1)^2$

$\bigcirc \ y=\left(\dfrac{-1}{3}\right) \cdot (x+1)^2-5$

b. Degree 3 polynomials or cubics

Standard form: $\quad y = a_3x^3 + a_2x^2 + a_1x + a_0,$

The "arms" of a degree 3 polynomial or cubic reach in opposite directions one upward towards infinity and the other downward toward negative infinity.
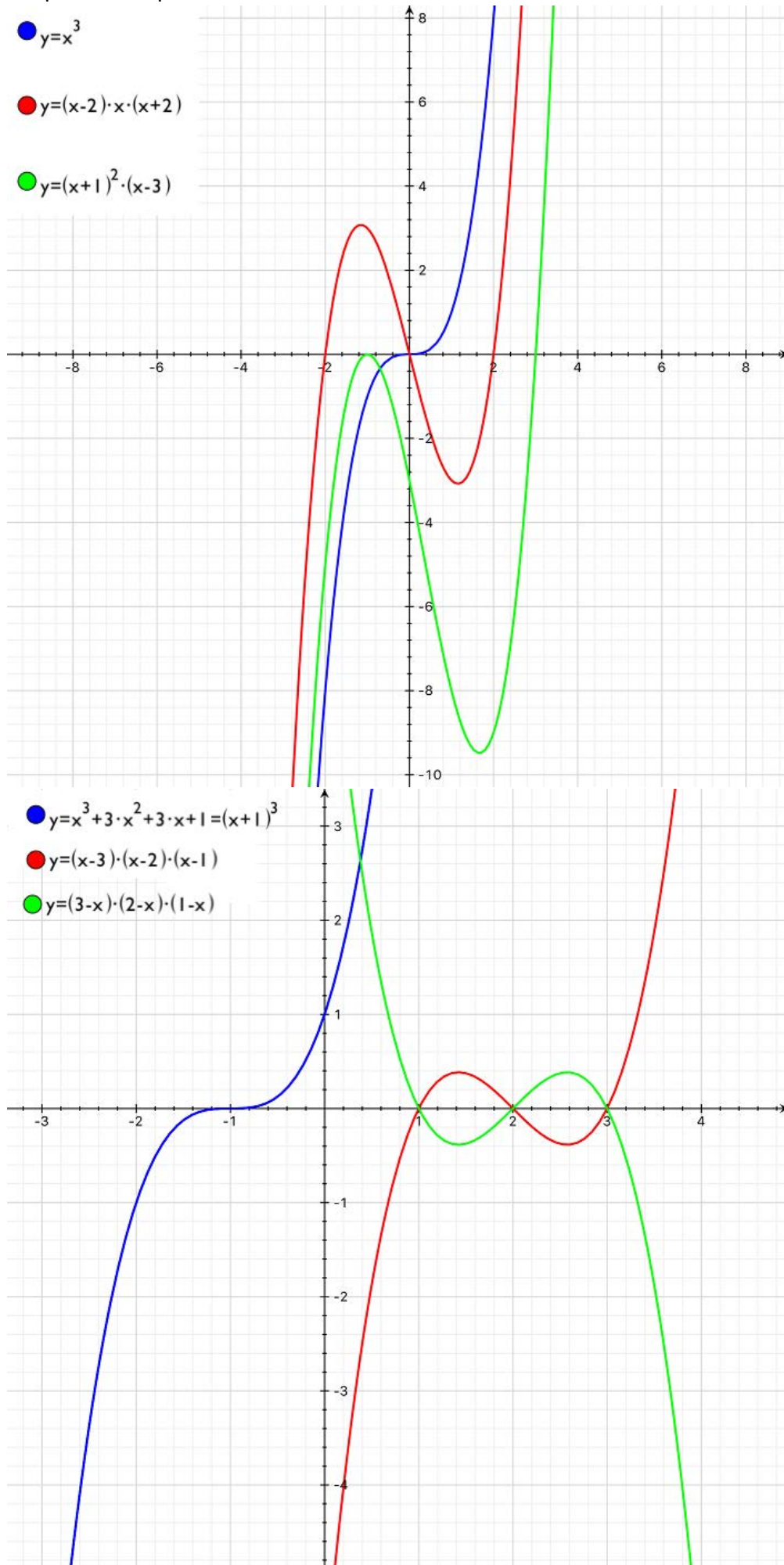
If a cubic only one distinct root then it will have no local extrema, otherwise it will have two local extrema one local minimum and one local maximum

When the roots of a cubic are not all distinct the curve changes (see below).

A cubic may have up to 3 real roots.

The range of a cubic is always $\mathbb{R}$.

Graphed Examples:

$y = x^3$

$y = (x-2) \cdot x \cdot (x+2)$

$y = (x+1)^2 \cdot (x-3)$



$y = x^3 + 3 \cdot x^2 + 3 \cdot x + 1 = (x+1)^3$

$y = (x-3) \cdot (x-2) \cdot (x-1)$

$y = (3-x) \cdot (2-x) \cdot (1-x)$

c. Degree 4 polynomials or quartic

Standard form: $\quad y = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$

The "arms" of a degree 4 polynomial or quartic reach in the same direction either both upward towards infinity or both downward towards negative infinity.
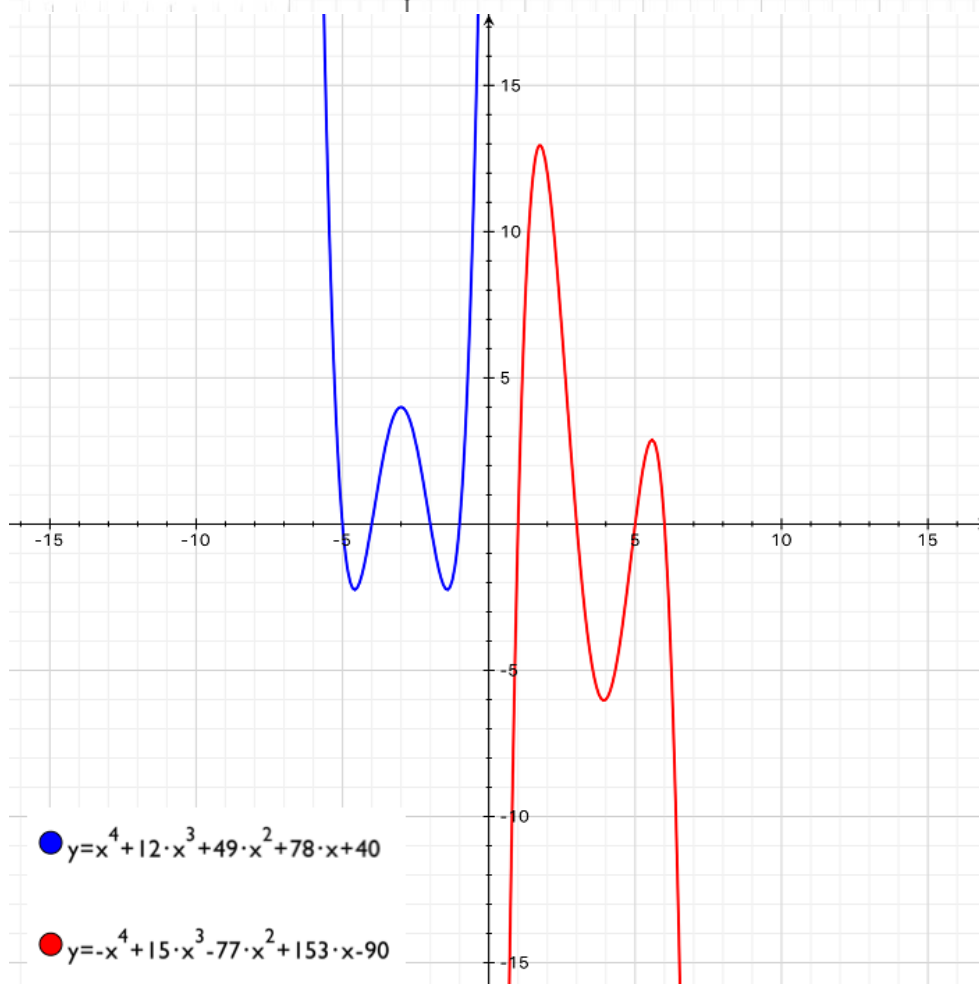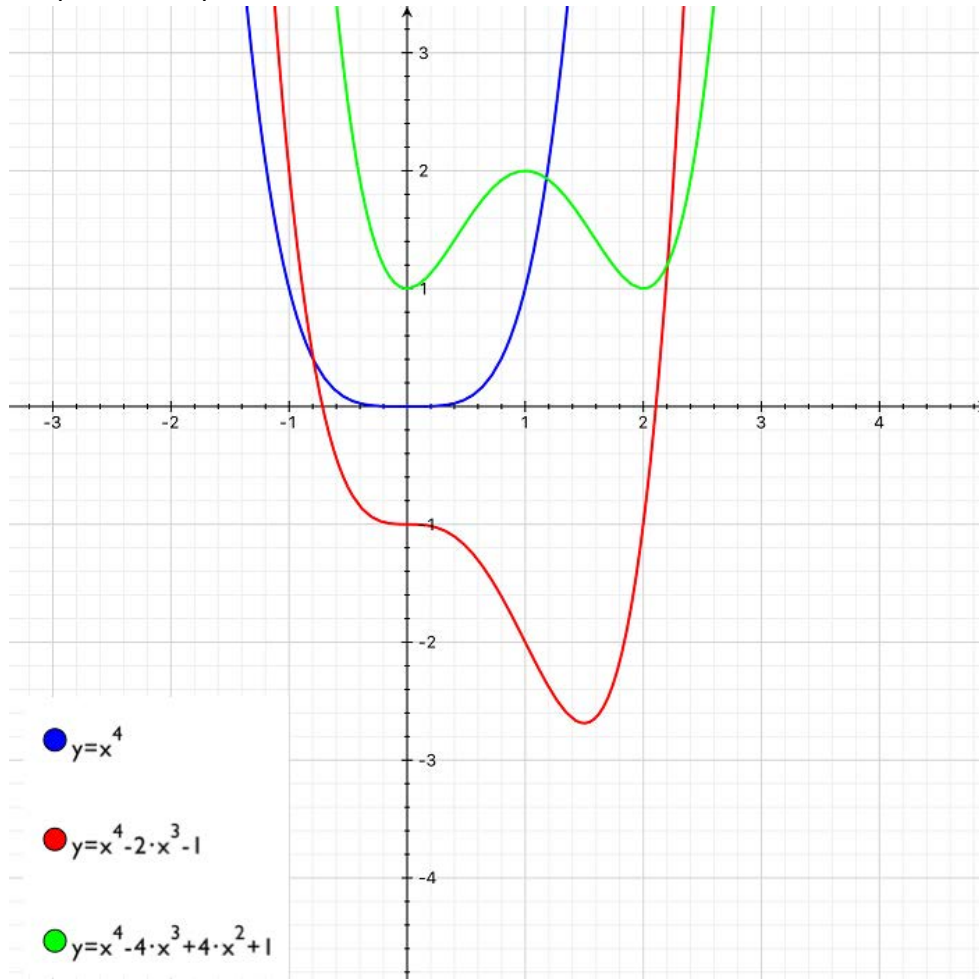
If a quartic only one distinct root then it will have one local extrema

When the roots of a quartic are all distinct the curve looks like an M or W, otherwise the curve changes (see below).

A quartic may have up to 4 real roots.

The range of a quartic depends on facing. The range is either the interval $[k, \infty)$ or $(-\infty, k]$ for some $k \in \mathbb{R}$.

Graphed Examples:



$y=x^4$

$y=x^4-2 \cdot x^3-1$

$y=x^4-4 \cdot x^3+4 \cdot x^2+1$



$y=x^4+12 \cdot x^3+49 \cdot x^2+78 \cdot x+40$

$y=-x^4+15 \cdot x^3-77 \cdot x^2+153 \cdot x-90$
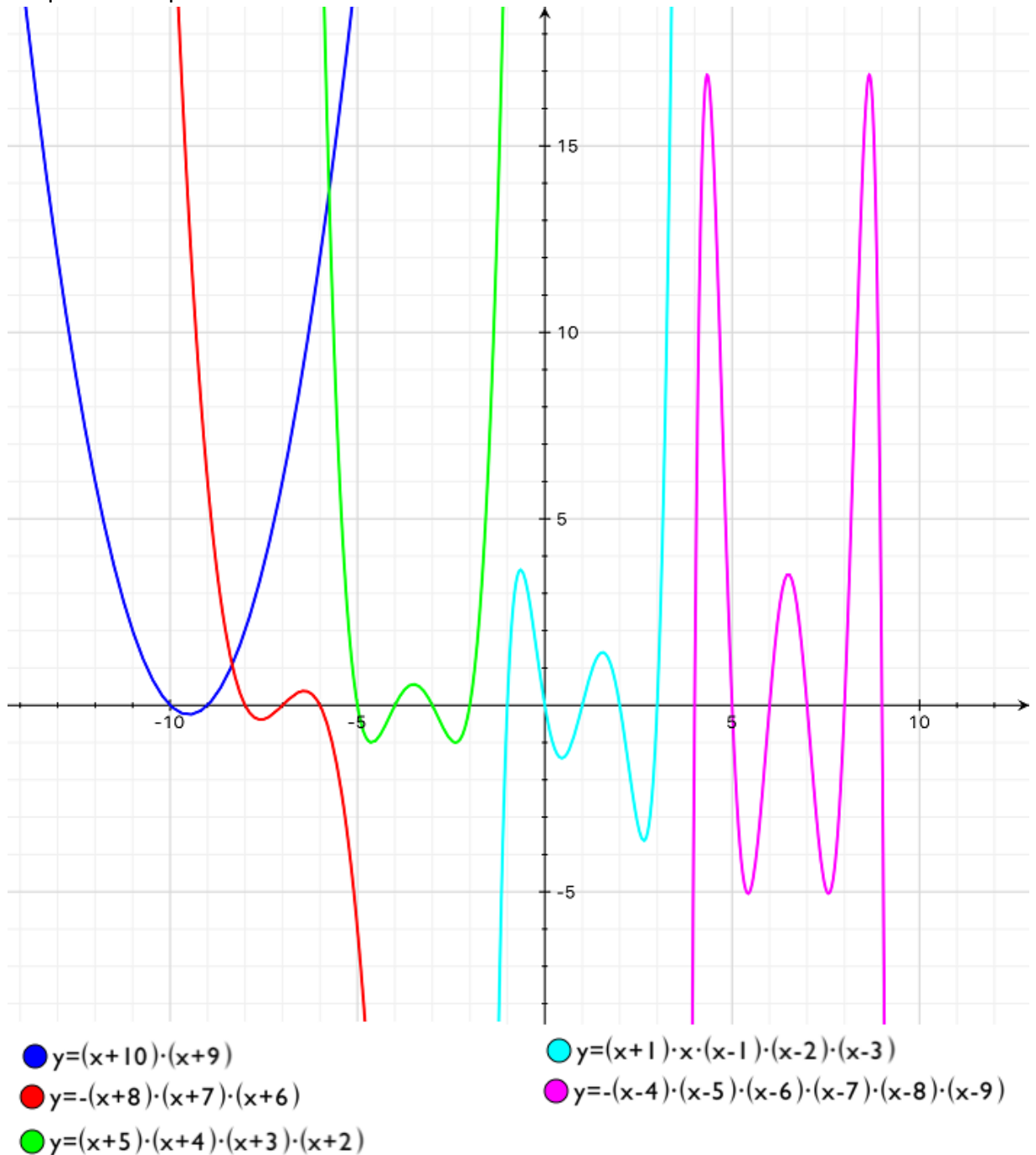
d. Degree $n$ polynomials

Standard form: $\quad y = a_n x^n + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0,$

If $n$ is odd then the "arms" of the polynomial reach in opposite directions one upward towards infinity and the other downward toward negative infinity and the range is $\mathbb{R}$.

If $n$ is even then the "arms" of the polynomial reach in the same direction either both upward towards infinity or both downward towards negative infinity and the range is either the interval $[k, \infty)$ or $(-\infty, k]$ for some $k \in \mathbb{R}$.

A degree $n$ polynomial can have up to $n - 1$ local extrema and up to $n$ real roots.

Graphed Examples:



● $y=(x+10)\cdot(x+9)$

● $y=-(x+8)\cdot(x+7)\cdot(x+6)$

● $y=(x+5)\cdot(x+4)\cdot(x+3)\cdot(x+2)$

● $y=(x+1)\cdot x\cdot(x-1)\cdot(x-2)\cdot(x-3)$

● $y=-(x-4)\cdot(x-5)\cdot(x-6)\cdot(x-7)\cdot(x-8)\cdot(x-9)$
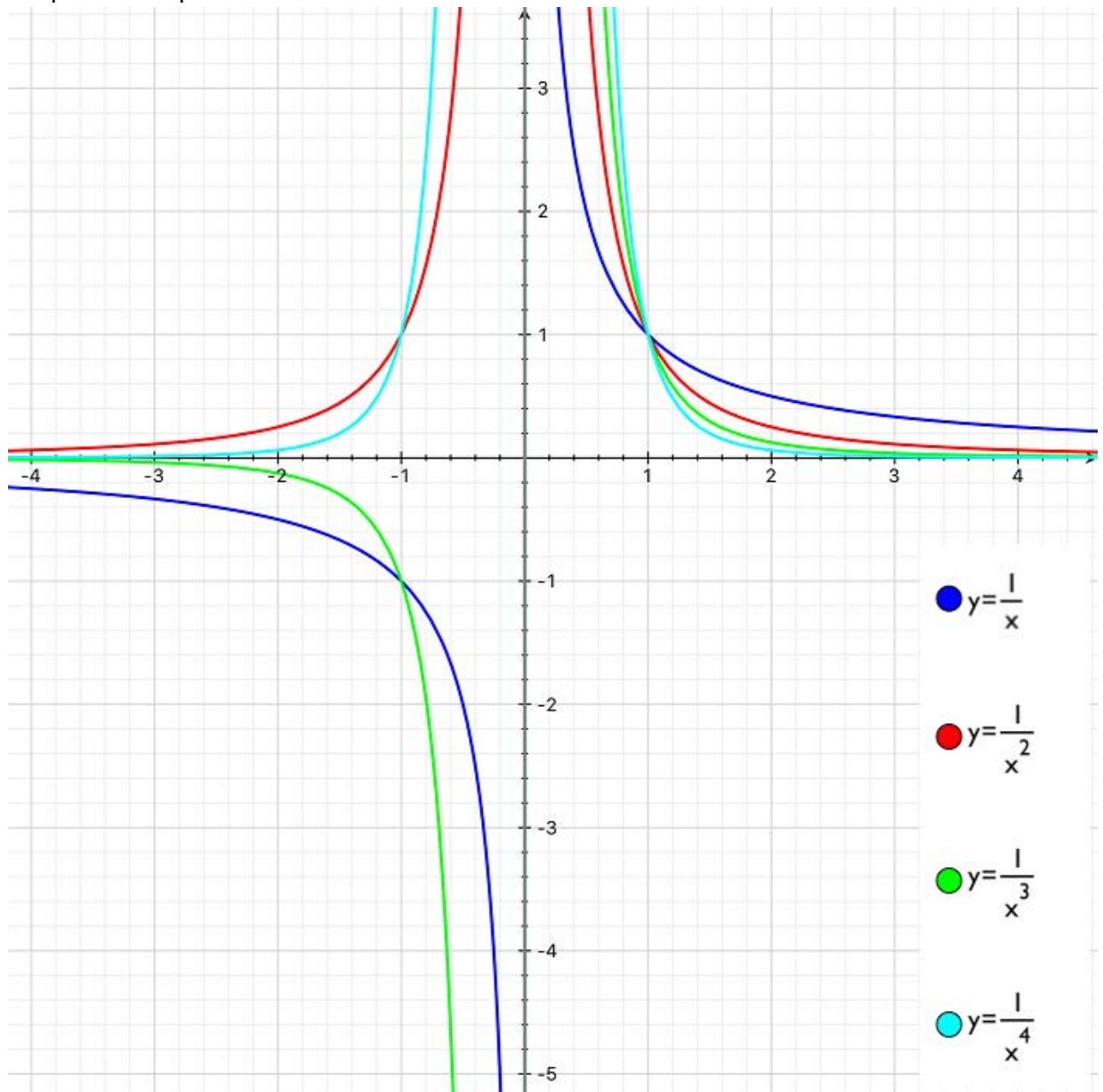
3. Simple rational functions

   a. The functions of the form $y = \frac{1}{x^a}, a \in \mathbb{Z}^+$ have a vertical asymptote at $x = 0$.
   The domain of this function is the interval $(-\infty, 0) \cup (0, \infty)$.
   If $a$ is even then the function stretches upward towards infinity on both sides of the $y$ −axis
   and the range is the interval $(0, \infty)$.
   If $a$ is odd then the function stretches upward towards infinity on the right side of the $y$ −axis
   and towards negative infinity on the left side of the $y$ −axis and the range is the interval
   $(-\infty, 0) \cup (0, \infty)$.
   Graphed Examples:

b.  The rational functions of the form $y = \frac{p(x)}{p(x)}$, where $p(x)$ is any polynomial are really just the line $y = 1$ with a hole in them anywhere that $p(x) = 0$.

It isn't much trickier if we throw an absolute value at the top or the bottom to get $y = \frac{|p(x)|}{p(x)} = \frac{p(x)}{|p(x)|}$. There is a hole when $p(x) = 0$ the line segments when $p(x) < 0$ are $y = -1$ and the line segments when $p(x) > 0$ are $y = 1$.
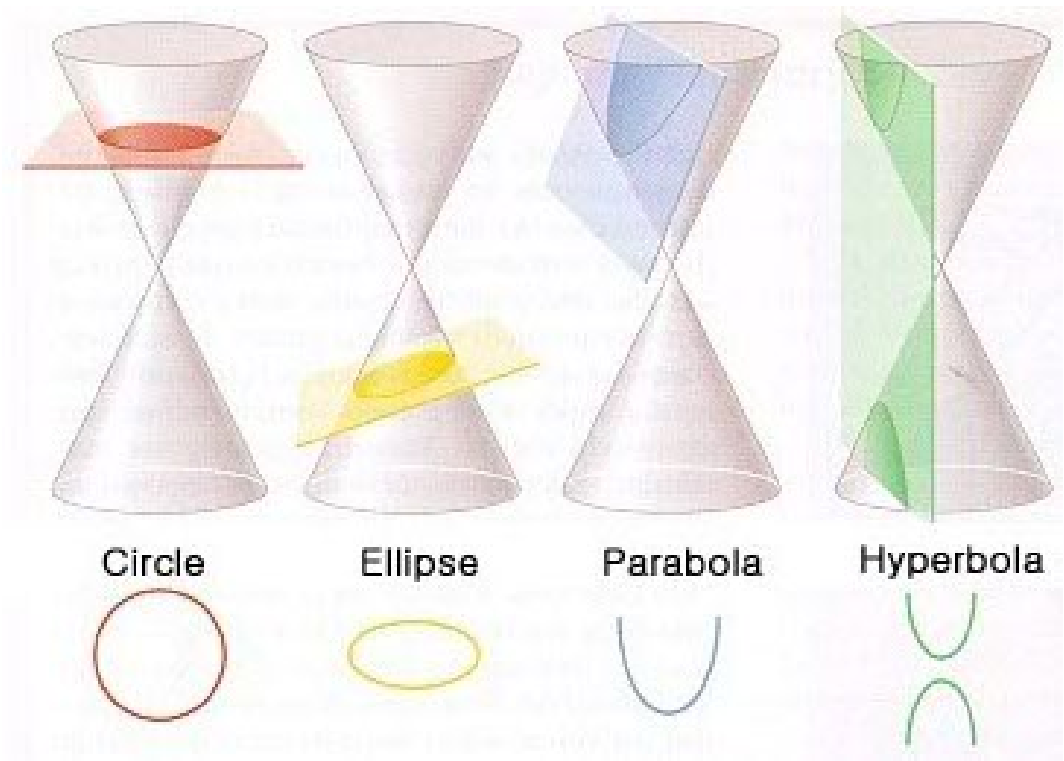
Graphed Examples:

$y = \frac{((x-1)\cdot(x+3))}{((x-1)\cdot(x+3))}$



$y = \frac{|x|}{x}$



$y = \frac{|(x-1)\cdot(x+2)\cdot(x-3)|}{(x-1)\cdot(x+2)\cdot(x-3)}$
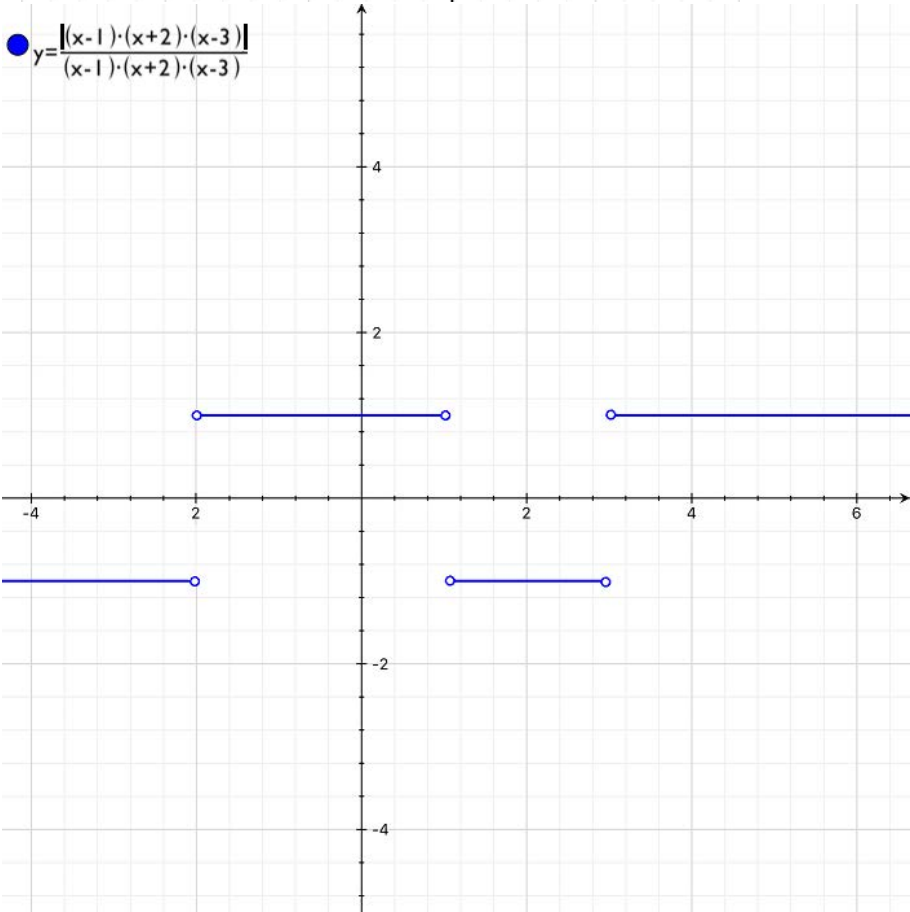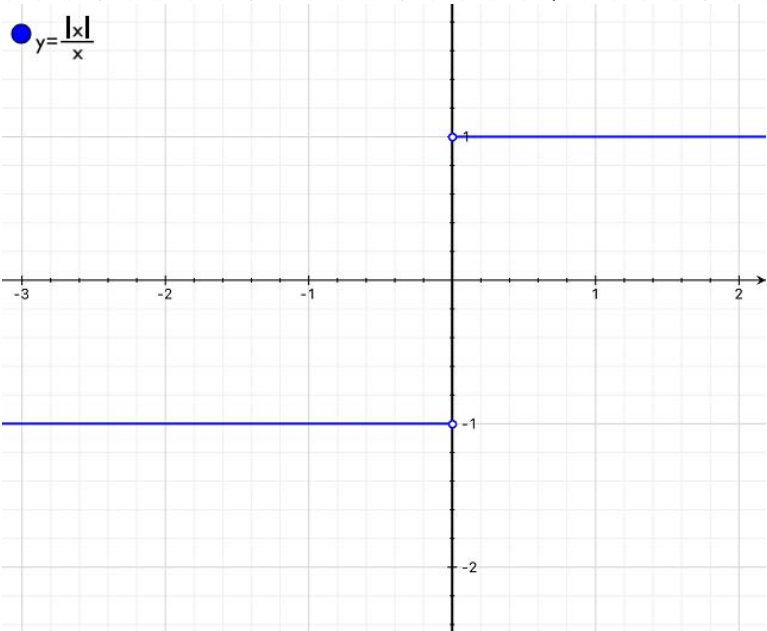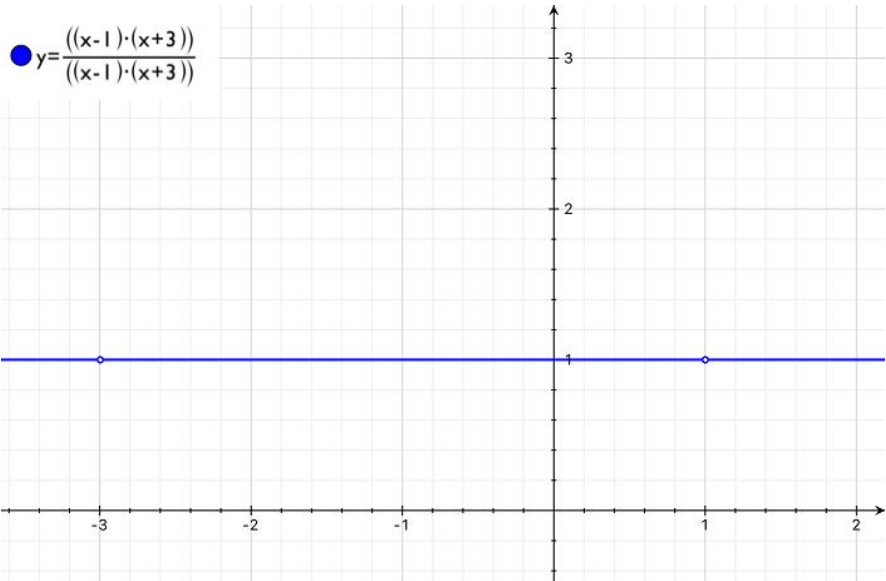
4. Conic Sections

These curves can be obtained by slicing a pair of cones and observing the edges, as illustrated below.



| Circle | Ellipse | Parabola | Hyperbola |

a. Circle

A circle of radius $r$ centred at $(h, k)$ has the formula $(x - h)^2 + (y - k)^2 = r^2$.
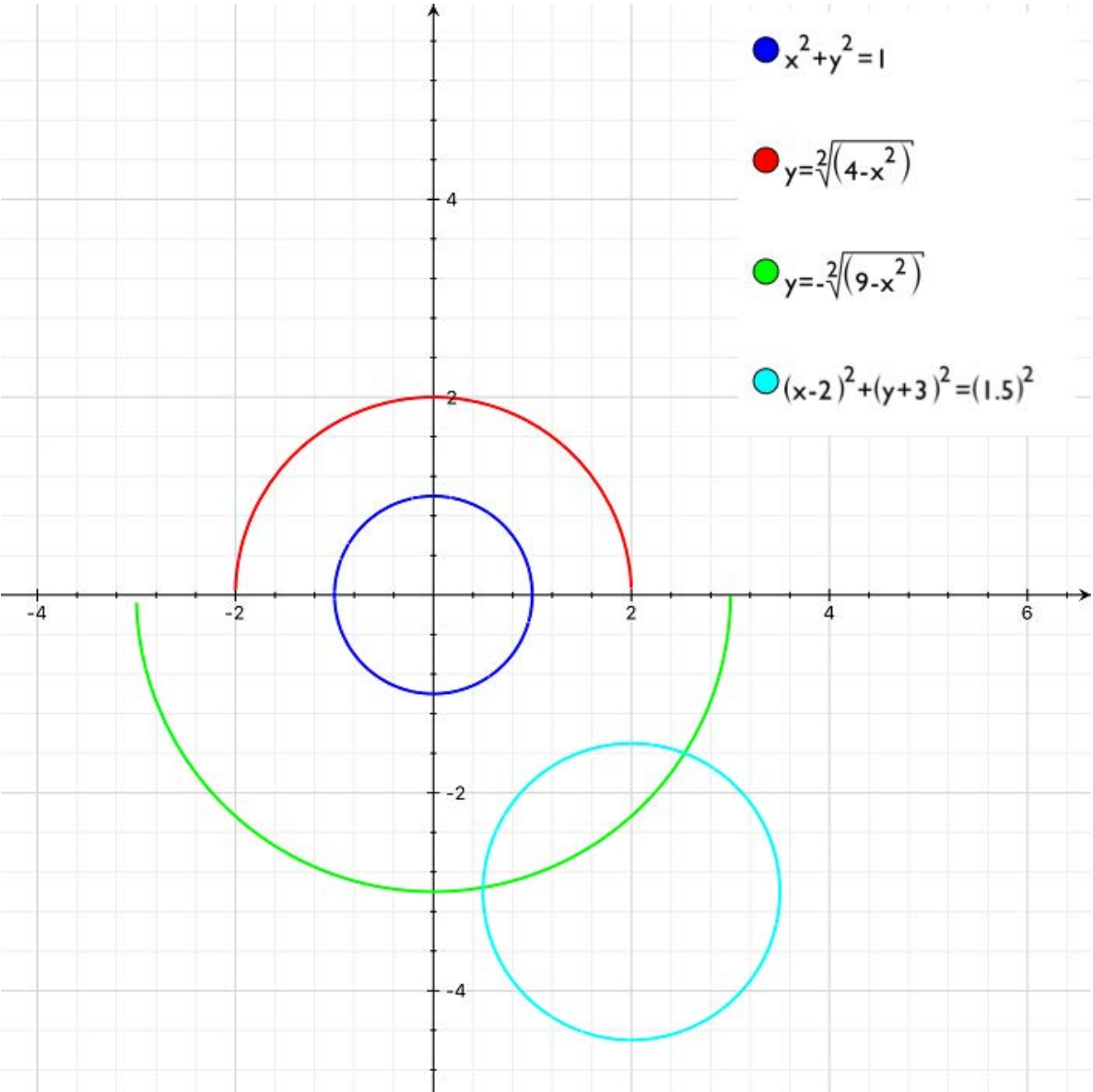
Notice that a circle is not a function (it fails the vertical line test) but both the upper half and lower half of a circle considered separately are functions.

The upper half of this circle is $y = k + \sqrt{r^2 - (x - h)^2}$

and the lower half of this circle is $y = k - \sqrt{r^2 - (x - h)^2}$.

The domain of a circle is the interval $[h - r, h + r]$ and the range is the interval $[k - r, k + r]$.

Graphed Examples:



$x^2 + y^2 = 1$

$y = \sqrt[2]{(4 - x^2)}$

$y = -\sqrt[2]{(9 - x^2)}$

$(x-2)^2 + (y+3)^2 = (1.5)^2$

b. Ellipse

An ellipse with a width of $2a$ and a height of $2b$ centered at the $(h, k)$ has the

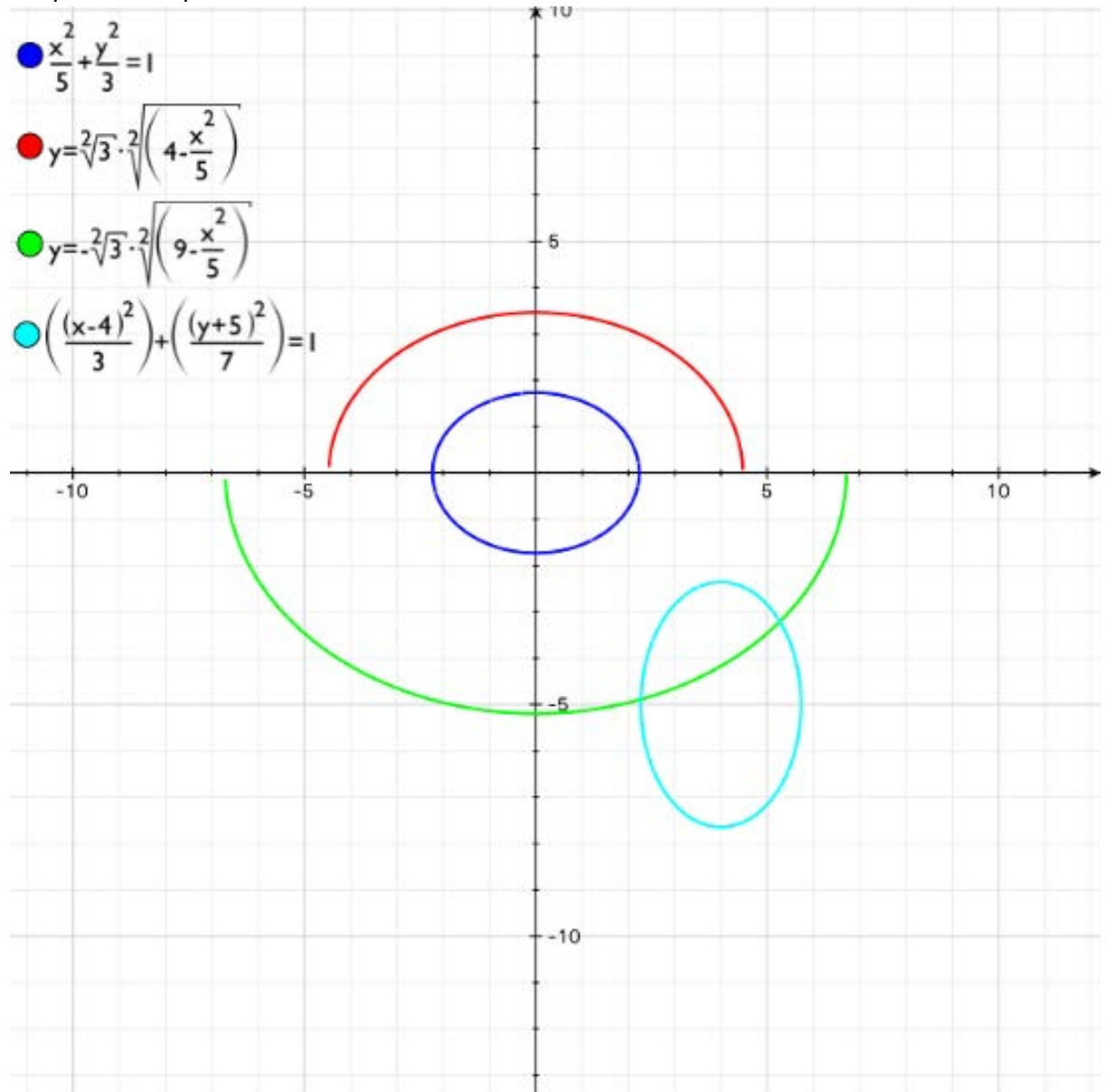formula $\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$

Notice that a ellipse is not a function (it fails the vertical line test) but both the upper half and lower half of an ellipse considered separately are functions.

The upper half of this ellipse is $y = k + b\sqrt{1 - \left(\frac{x-h}{a}\right)^2}$

and the lower half of this ellipse is $y = k - b\sqrt{1 - \left(\frac{x-h}{a}\right)^2}$

The domain of an ellipse is the interval $[h - a, h + a]$ and the range is the interval $[k - b, k + b]$.

Graphed Examples:

● $\frac{x^2}{5} + \frac{y^2}{3} = 1$

● $y = \sqrt[2]{3} \cdot \sqrt{2\left(4 - \frac{x^2}{5}\right)}$

● $y = -\sqrt[2]{3} \cdot \sqrt{2\left(9 - \frac{x^2}{5}\right)}$

○ $\left(\frac{(x-4)^2}{3}\right) + \left(\frac{(y+5)^2}{7}\right) = 1$



c. Parabola

Again?!? – No, not really, nothing has changed since page 1 but I couldn't discuss conic sections without mentioning that the parabola is one of them.

d. Hyperbola

A hyperbola consists of two symmetric curves that each somewhat resemble parabolae.

A hyperbola with the formula $\left(\frac{x-h}{a}\right)^2 - \left(\frac{y-k}{b}\right)^2 = 1$ or $\left(\frac{y-k}{b}\right)^2 - \left(\frac{x-h}{a}\right)^2 = 1$ is centred at the point $(h, k)$. It is symmetric around both the line $x = h$ and the line $y = k$.

The hyperbola $\left(\frac{x-h}{a}\right)^2 - \left(\frac{y-k}{b}\right)^2 = 1$ can be described as an east-west hyperbola.

The hyperbola $\left(\frac{y-k}{b}\right)^2 - \left(\frac{x-h}{a}\right)^2 = 1$ can be described as an north-south hyperbola.

The lines $y = \frac{a}{b}(x - h) + k$ and $y = -\frac{a}{b}(x - h) + k$ are asymptotes for the hyperbola.

Hyperbolae are not functions because they fail the vertical line test.

The domain and range of an east-west hyperbola are $(-\infty, h - a]\cup[h + a, \infty)$ and $\mathbb{R}$ respectively.

The domain and range of a north-south hyperbola are $\mathbb{R}$ and $(-\infty, k - b]\cup[k + b, \infty)$ respectively.
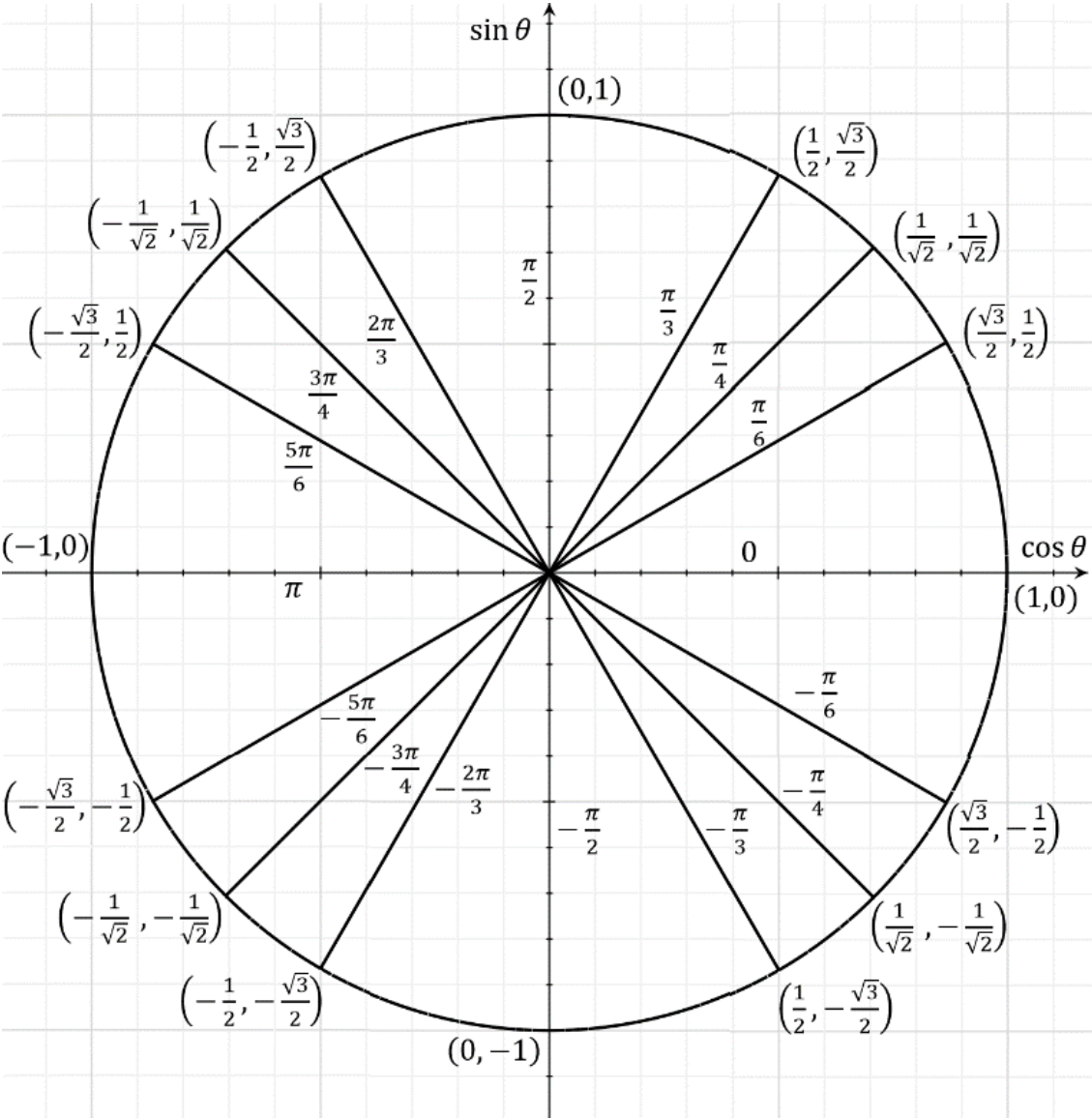
Graphed Examples:
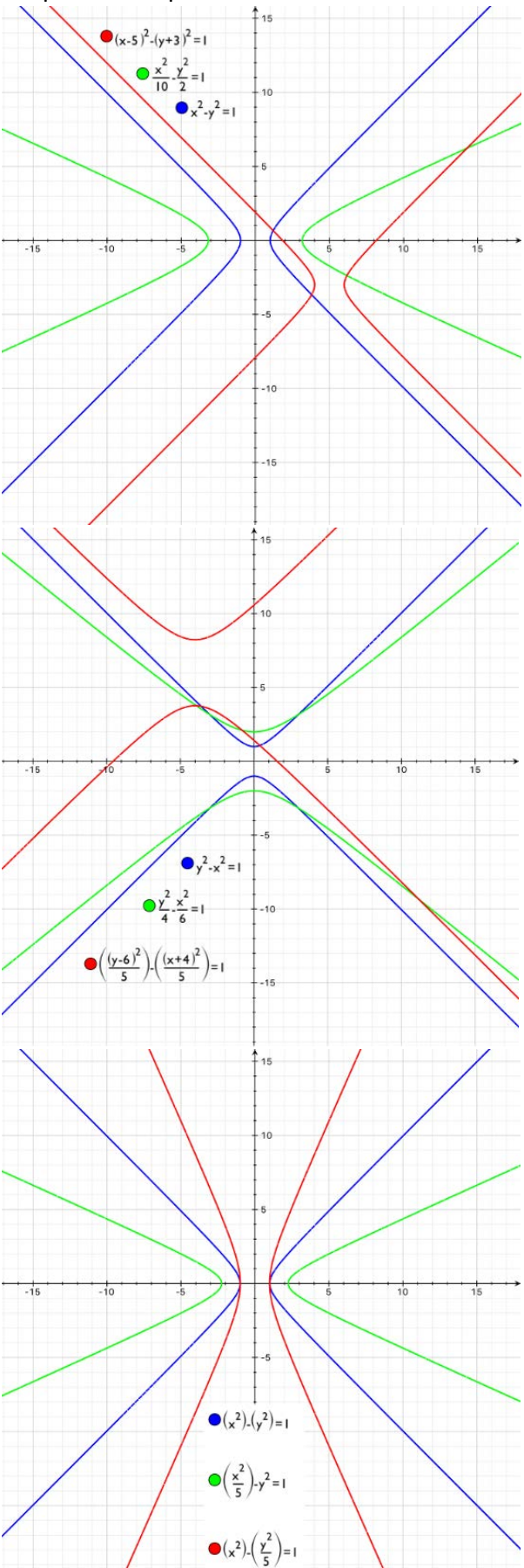
5. Trigonometric functions

In University you will almost always be using radians rather than degrees when using trig functions. Recall that a circle contains $2\pi$ radians and a right angle is $\frac{\pi}{2}$ radians.

You will be expected to have the sine and cosine of all the standard angles memorized.
Many people use the unit circle to help them remember the sine and cosine of the standard angles.
This is a unit circle:

$\sin\theta$

$(0,1)$

$\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ $\quad\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

$\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ $\quad\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

$\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ $\quad\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

$\frac{\pi}{2}$ $\quad 2\pi/3$ $\quad \frac{\pi}{3}$ $\quad \frac{\pi}{4}$ $\quad \frac{3\pi}{4}$ $\quad \frac{\pi}{6}$ $\quad \frac{5\pi}{6}$

$(-1,0)$ $\qquad 0 \qquad \cos\theta$

$\pi$ $\qquad (1,0)$

$-\frac{5\pi}{6}$ $\quad -\frac{\pi}{6}$

$-\frac{3\pi}{4}$ $\quad -\frac{2\pi}{3}$ $\quad -\frac{\pi}{4}$

$\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$ $\qquad -\frac{\pi}{2} \quad -\frac{\pi}{3}$ $\qquad \left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$

$\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ $\qquad \left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

$\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ $\qquad \left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

$(0,-1)$

When solving something like $a = \sin x$ for some constant $a \in \mathbb{R}$ be sure to give **ALL** the appropriate answers.

For example: $\sin x = \frac{1}{\sqrt{2}}$ when $x = \pm\frac{\pi}{4} + 2\pi n$, for all $n \in \mathbb{Z}$.

a. Sine

The function $y = \sin x$ is a wave with a period of $2\pi$.
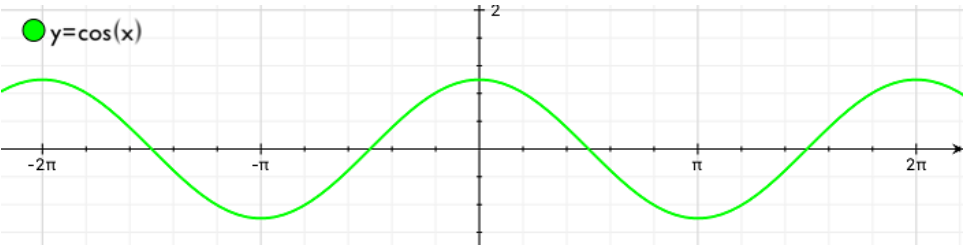The domain of sine is $\mathbb{R}$ and the range is $[-1,1]$.
Graphing Example:
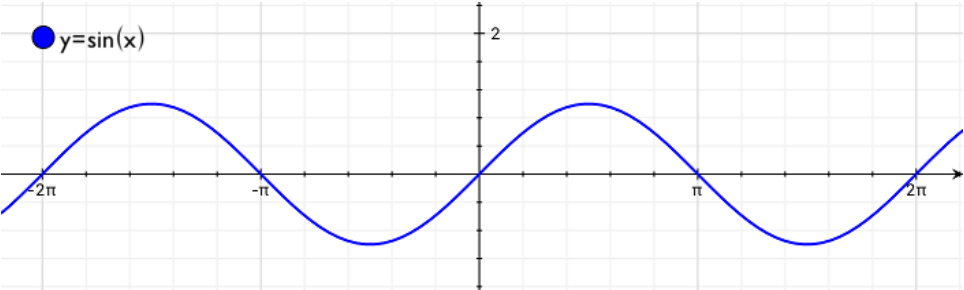


b. Cosine

The function $y = \cos x$ is a wave with a period of $2\pi$.
The domain of cosine is $\mathbb{R}$ and the range is $[-1,1]$.
As you may notice by observing their graphs cosine is simply the sine curve shifted left $\frac{\pi}{2}$.
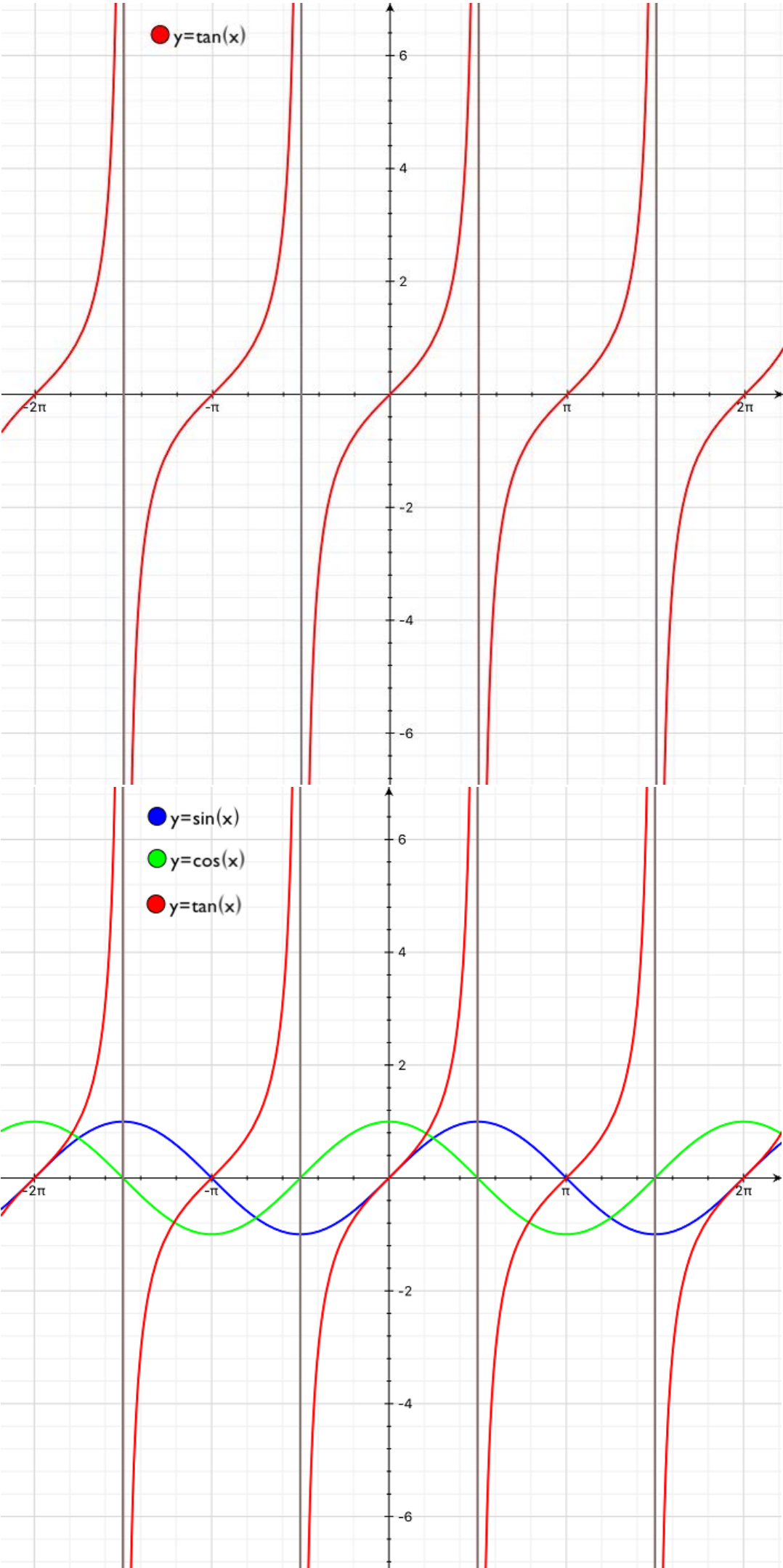Graphing Example:

c. Tangent

The function $y = \tan x$ is a function with a period of $\pi$ and vertical asymptotes at $x = \frac{\pi}{2} + n\pi$, for all $n \in \mathbb{Z}$.

The domain of tangent is $\mathbb{R} - \{x \in \mathbb{R}: x = \frac{\pi}{2} + n\pi$, for all $n \in \mathbb{Z}\}$ and the range is $\mathbb{R}$.

Recall $\tan x = \frac{\sin x}{\cos x}$. This relationship is illustrated in the graph below.
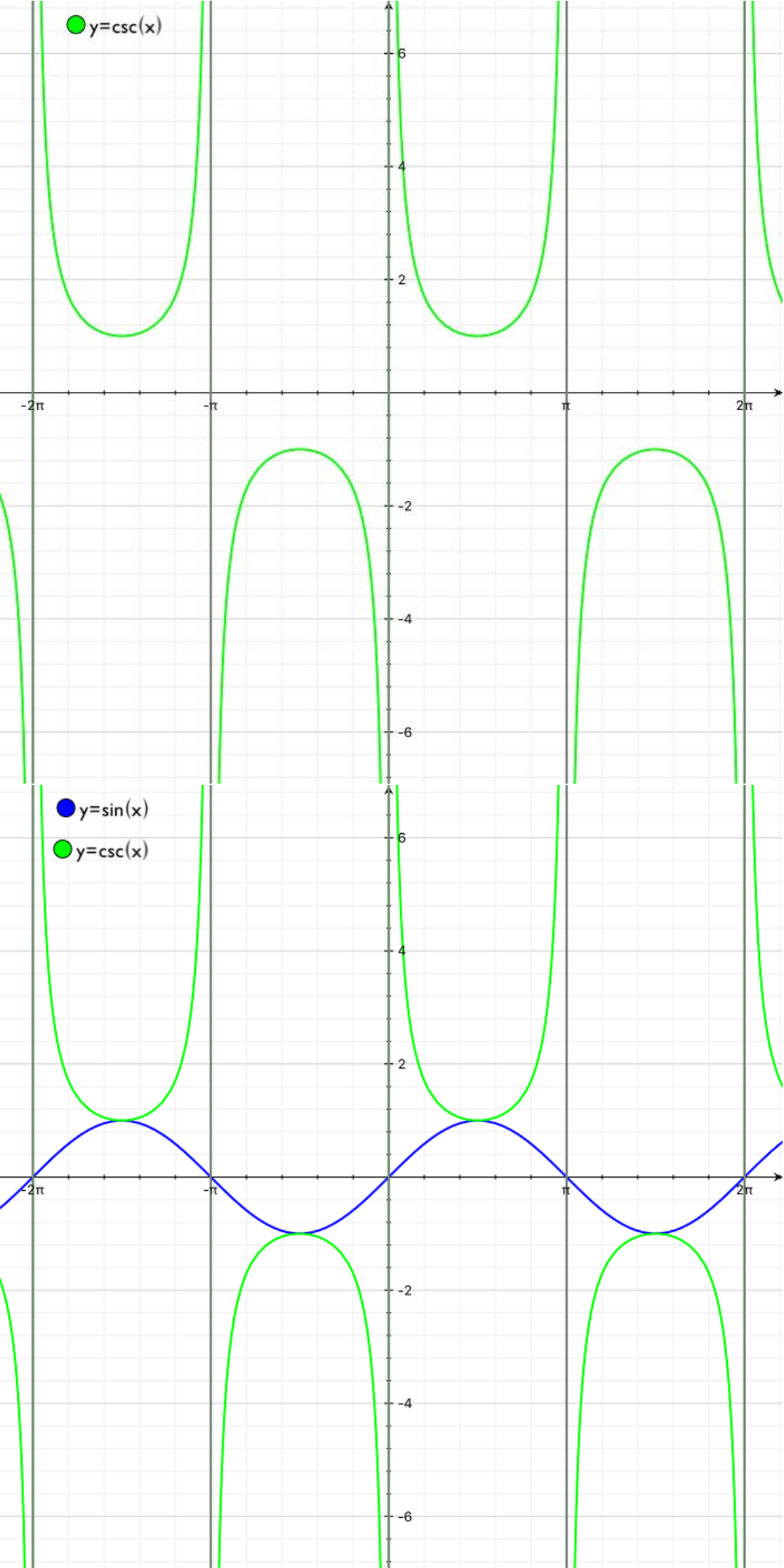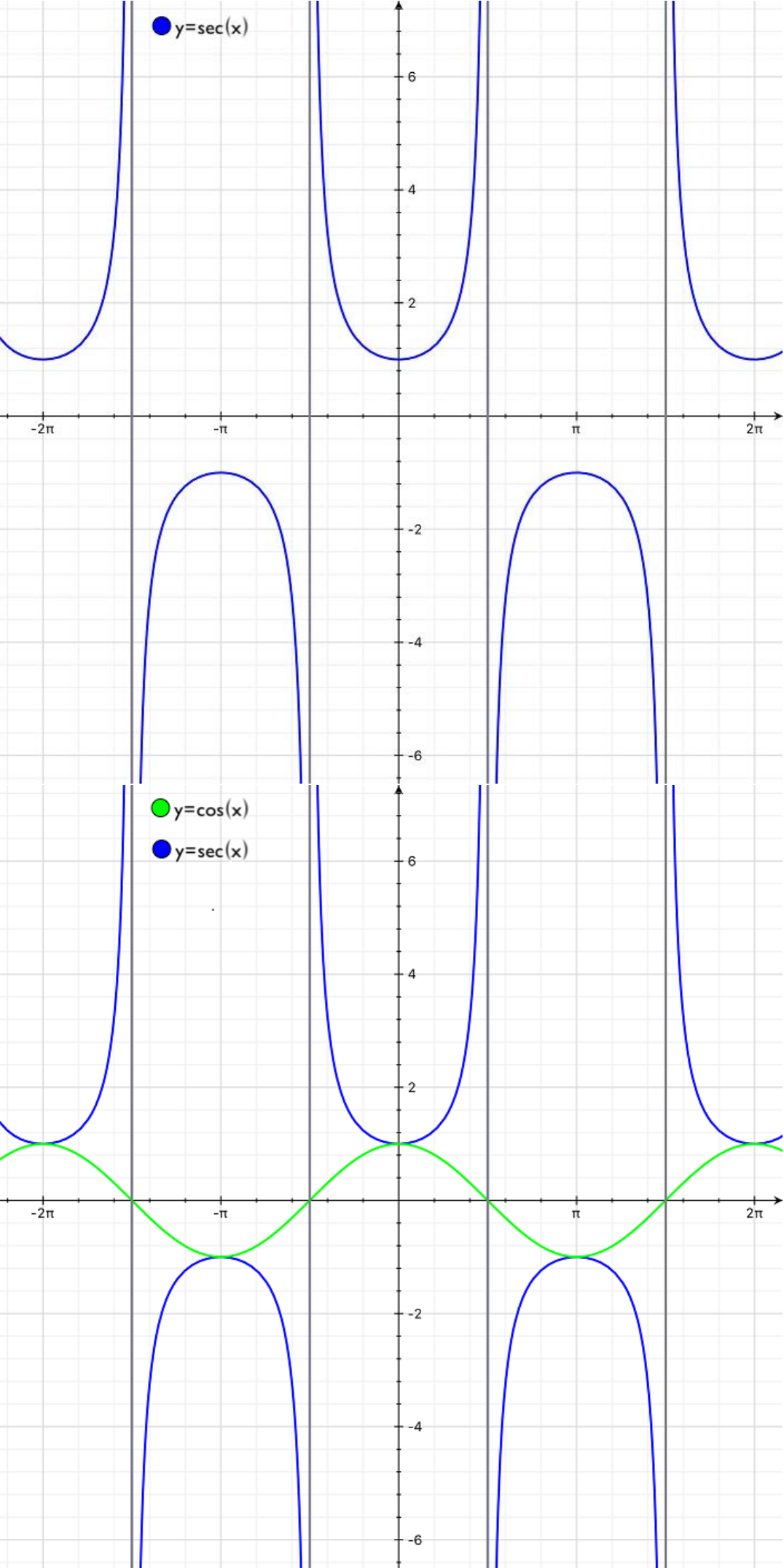
Graphing Examples:

d. Cosecant

The function $y = \csc x$ is a is a function with a period of $2\pi$ and vertical asymptotes at $x = n\pi$, for all $n \in \mathbb{Z}$.

The domain of sine is $\mathbb{R} - \{x \in \mathbb{R} : x = \pi n, \text{ for all } n \in \mathbb{Z}\}$ and the range is $(-\infty, -1] \cup [1, \infty)$.

Recall $\csc x = \dfrac{1}{\sin x}$. This relationship is illustrated in the graph below.

Graphing Example:

e. Secant

The function $y = \sec x$ is a is a function with a period of $2\pi$ and vertical asymptotes at $x = \frac{\pi}{2} + n\pi$, for all $n \in \mathbb{Z}$.

The domain of sine is $\mathbb{R} - \{x \in \mathbb{R} : x = \frac{\pi}{2} + \pi n$, for all $n \in \mathbb{Z}\}$ and the range is $(-\infty, -1] \cup [1, \infty)$.

Recall $\sec x = \frac{1}{\cos x}$. This relationship is illustrated in the graph below.

Graphing Examples:

f. Cotangent

The function $y = \cot x$ is a function with a period of $\pi$ and vertical asymptotes at $x = n\pi$, for all $n \in \mathbb{Z}$.

The domain of cotangent is $\mathbb{R} - \{x \in \mathbb{R}: x = n\pi, \text{ for all } n \in \mathbb{Z}\}$ and the range is $\mathbb{R}$.

Recall $\cot x = \dfrac{\cos x}{\sin x}$. This relationship is illustrated in the graph below.

Graphing Examples:
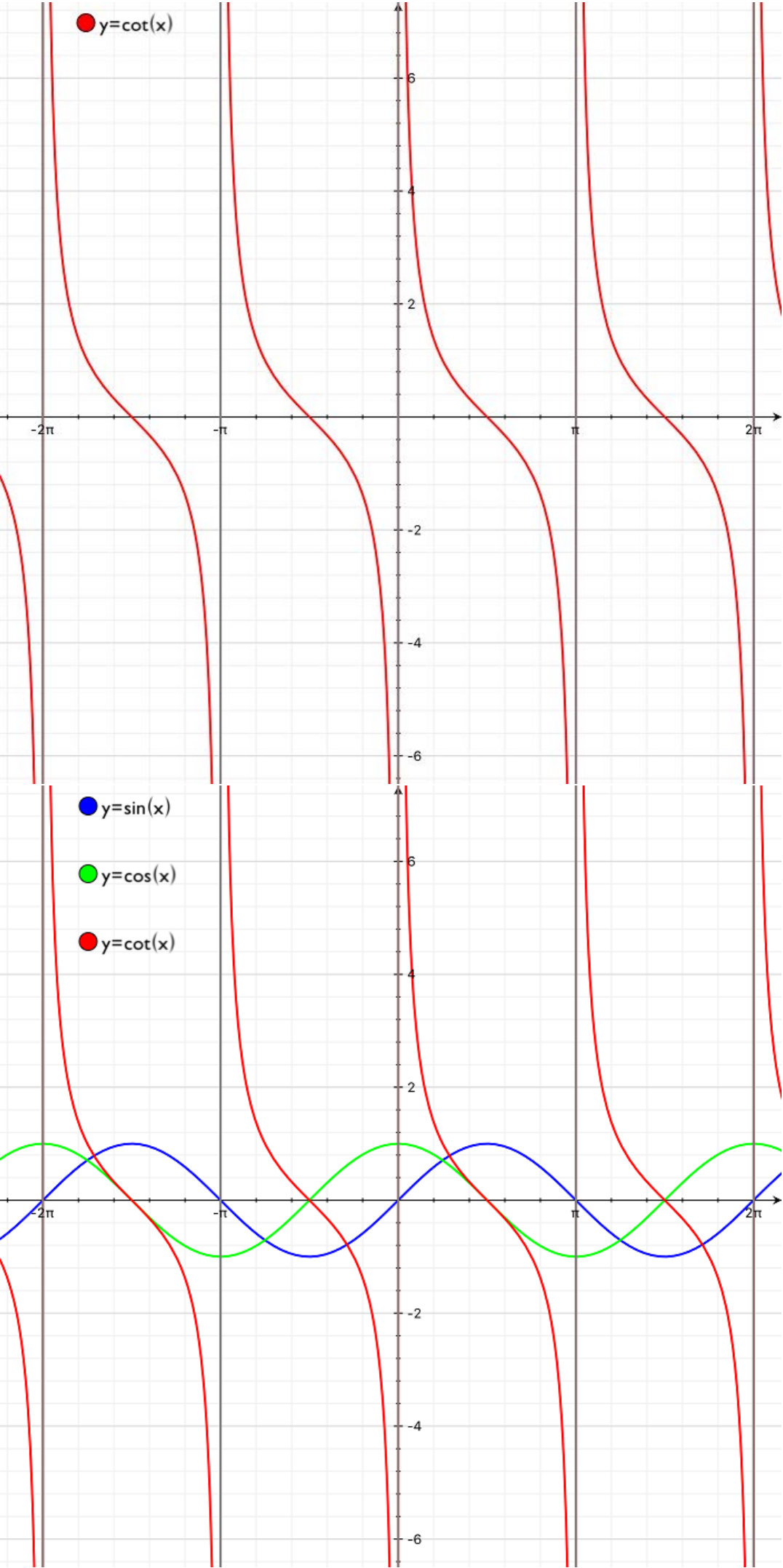
6. Exponential functions

Consider the exponential functions having the form $y = a^x$ for constant, $a \in (0,1)\cup(1,\infty)$.

Note: We do not wish to consider $a < 0$ because the function would be discontinuous at every even exponent and undefined for every exponent that is an even root. We do not consider $a = 1$ because $1^x = 1$ for all $x \in \mathbb{R}$ and that is just the line $y = 1$.

The domain of exponential functions, $y = a^x$, is $\mathbb{R}$ and the range is $(0, \infty)$.

All exponential functions of the form $y = a^x$ pass through the point $(0,1)$, sweep up towards infinity on the right side of the $y$ −axis, and get closer and closer to the $x$ −axis as they extend to the left.

Graphing Example:
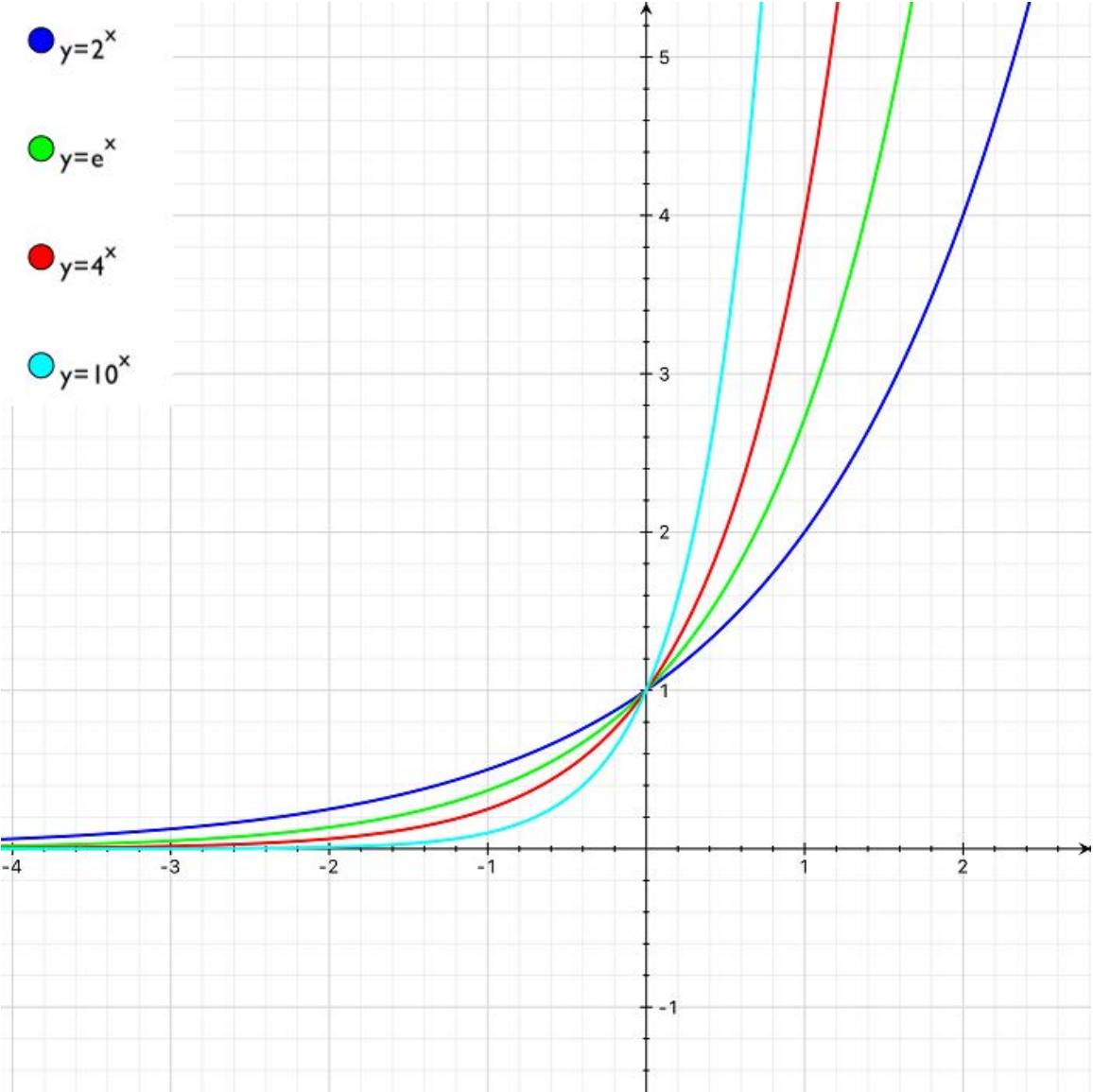
7. Logarithmic functions

Consider the logarithmic functions having the form $y = \log_a x$ for constant, $a \in (0,1) \cup (1, \infty)$.

Notice that $y = \log_a x$ if and only if $a^y = x$.

Thus we inherit useful values for $a$ from the exponential function.

The domain of logarithmic function, $y = \log_a x$, is $(0, \infty)$ and the range is $\mathbb{R}$.

All logarithmic functions of the form $y = \log_a x$ pass through the point $(1,0)$, drift up towards infinity on the at the right end of the curve, and drop toward negative infinity as $x$ values get closer to the $y-$axis.

There is a special logarithmic function $y = \ln x = \log_e x$ where $e$ is a real number with an infinite non-repeating decimal expansion, $e \cong 2.7182818284 \ldots$ (To learn more about $e$ and it's importance take lots and lots of fun math classes :-D )

Graphing Example:



$\bullet\ y = \log_{\frac{1}{2}} x$

$\bullet\ y = \log_2 x$

$\bullet\ y = \log_e x = \ln x$

$\circ\ y = \log_4 x$
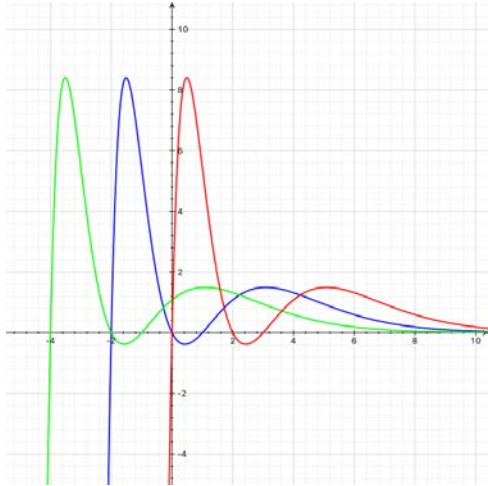
$\bullet\ y = \log_{10} x$
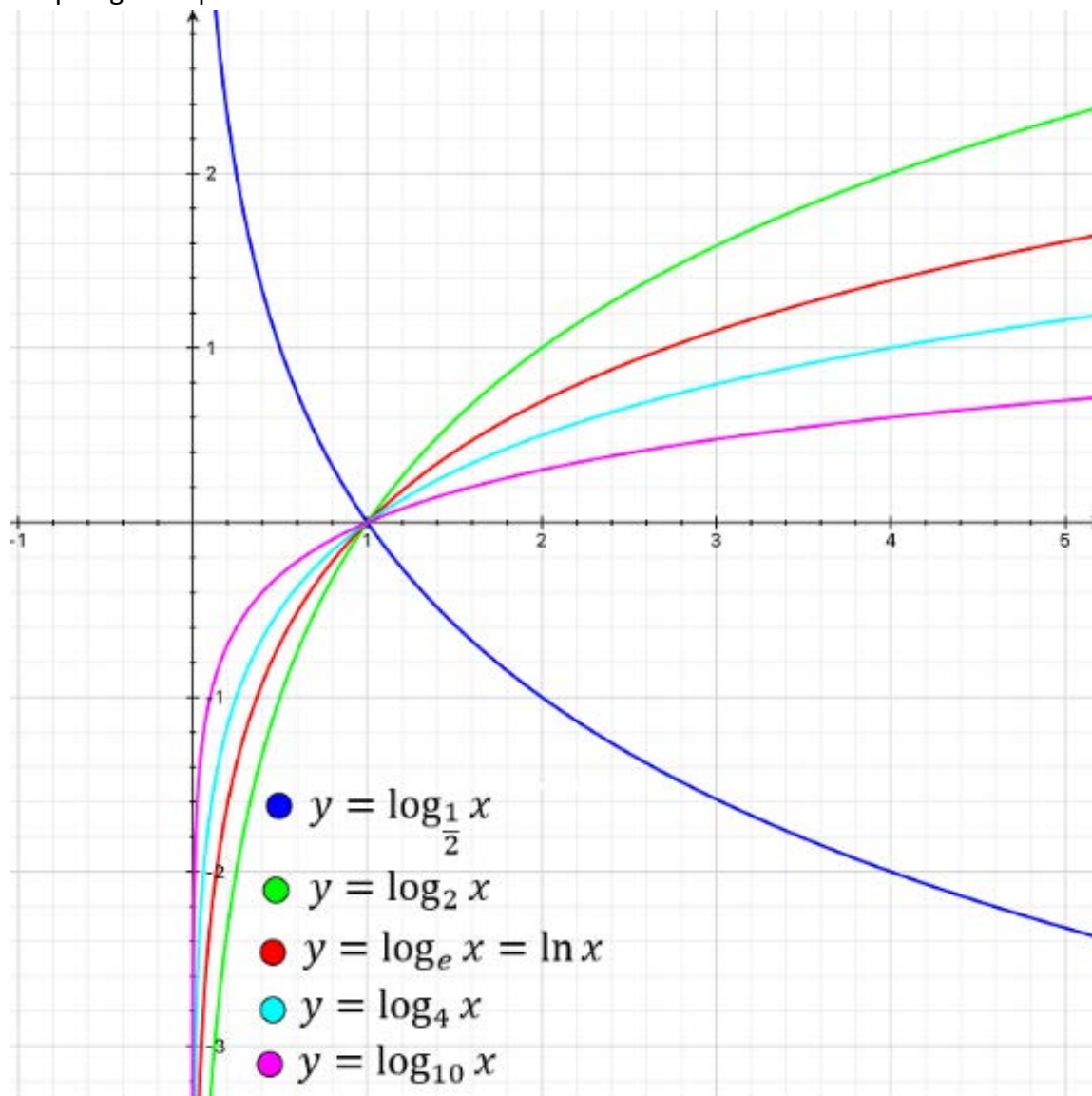
8. Shifts and transformations

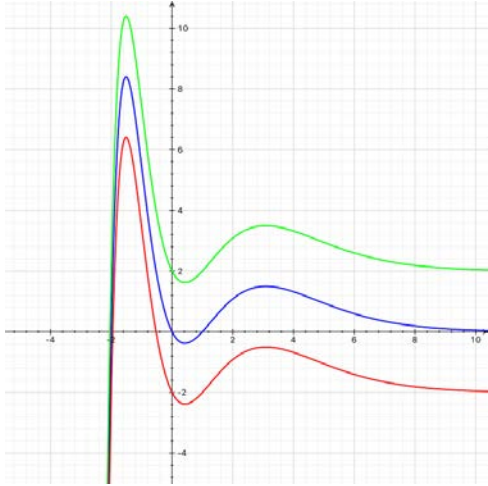For any curve $y = f(x)$ and $a \in (0, \infty)$ and $b \in (1, \infty)$
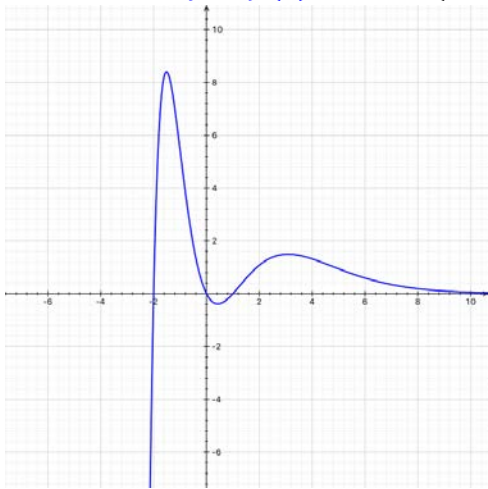
$y = f(x + a)$ shifts the curve $a$ units to the left.
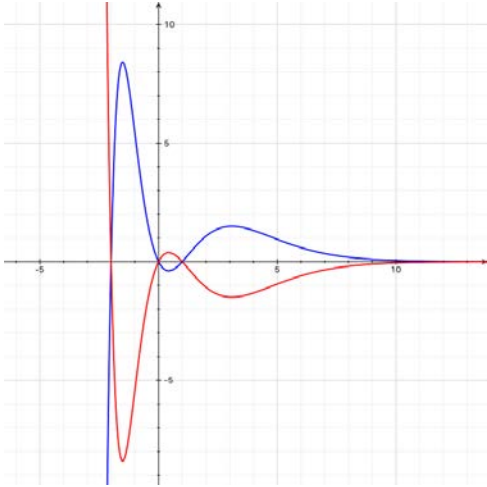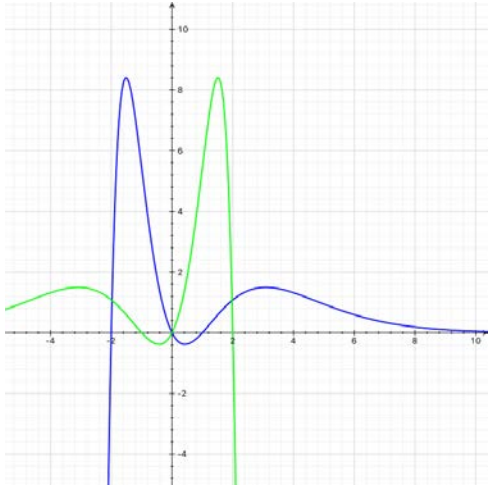$y = f(x - a)$ shifts the curve $a$ units to the right.

$y = f(x) + a$ shifts the curve $a$ units up.
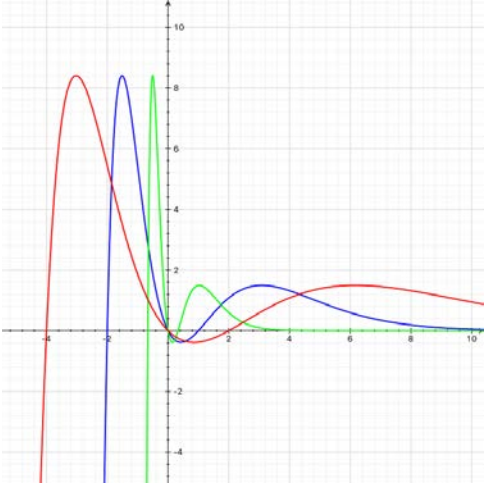$y = f(x) - a$ shifts the curve $a$ units down.

$y = f(-x)$ reflects the curve across the $y-$axis.
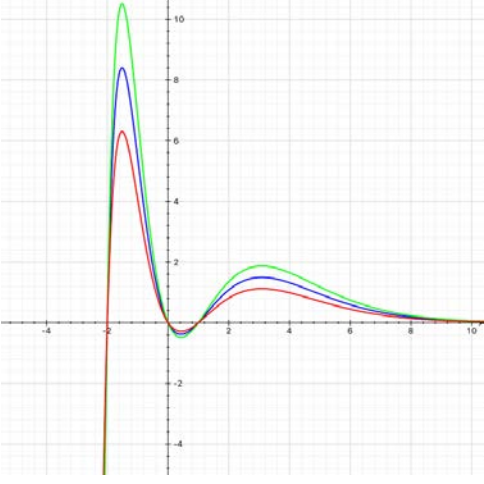$y = -f(x)$ reflects the curve across the $x-$axis.

$y = f(bx)$ will squeeze the curve in a left-right direction.

$y = f\left(\frac{1}{b}x\right)$ will stretch the curve in a left-right direction.
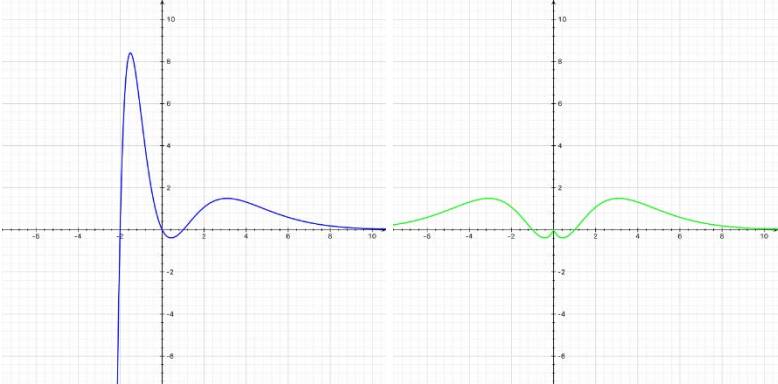
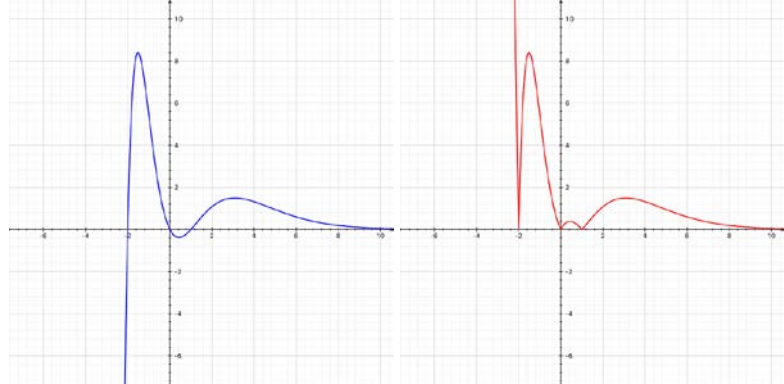$y = bf(x)$ will stretch the curve in a north-south direction.

$y = \frac{1}{b}f(x)$ will squeeze the curve in a north-south direction.

$y = f(|x|)$ will change the curve left of the $y$ −axis to the reflection of the curve on the right of the $y$ −axis.

$y = |f(x)|$ will fold the curve upward across the $x$ −axis.

$x = f(y)$ will reflect the curve around the line $y = x$.