On the Hardness of Turn-Angle-Restricted Rectilinear Cycle Cover Problems

Extended Abstract

Steph Durocher* and David Kirkpatrick†
Department of Computer Science, University of British Columbia
Vancouver BC, Canada

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Abstract

A cycle cover of a graph G is a collection of disjoint cycles that spans G. Generally, a (possibly disconnected) cycle cover is easier to construct than a connected (Hamiltonian) cycle cover. One might expect this since the cycle cover property is local whereas connectivity is a global constraint. We compare the hardness of CONNECTED CYCLE COVER and CYCLE COVER under various constraints (both local and global) on the orientation, crossings, and turning angles of edges. Surprisingly perhaps, under specific constraints, the cycle cover problem is NP-hard whereas the corresponding connected cycle cover problem can be solved in polynomial time.

1 Definitions

A *straight-line embedding* of graph $G = (V, E)$ is a one-to-one mapping of vertices to the plane, $\delta : V \to \mathbb{Z}^2$, where every edge $(u, v) \in E$ maps to the unique line segment between $\delta(u)$ and $\delta(v)$. $\hat{G} = (V, E, \delta)$ is the corresponding *geometric graph.* Two edges *cross* if and only if they share an interior point. A geometric graph $\hat{G}$ is *non-crossing* if no two of its edges cross.

A set S of non-degenerate line segments is an *orthogonal* set if there exists an angle ϕ such that all segments in S form an angle congruent to $\phi \mod \frac{\pi}{2}$. An orthogonal set is *standard* if $\phi \mod \frac{\pi}{2} = 0$.

A geometric graph $\hat{G}$ is *(standard) rectilinear* if its edge set is (standard) orthogonal. Embedded graph $\hat{G}$ is *weakly rectilinear* if each of its connected components is rectilinear. Obviously, weak rectilinearity is the same as rectilinearity if $\hat{G}$ is connected.

A *cycle cover* (or *2-factor*) of a graph G consists of a 2-regular spanning subgraph $H \subseteq G$ (i.e. a union of cycles in which every vertex is incident with exactly two edges).

Given a cycle cover $\hat{H}$ of a geometric graph $\hat{G} = (V, E, \delta)$, the edges incident with a vertex $v_i \in V$ in $\hat{H}$ form an acute angle $\theta_i \in (0, \pi]$ (refered to as the *turn angle*) at v_i. Let Θ be the set of all turn angles θ_i formed at vertices $v_i \in V$. Let $\Psi \subseteq (0, \pi]$ be a set of angles. $\hat{H}$ is Ψ*-turn-restricted* if $\Theta \subseteq \Psi$.

While a $\{\frac{\pi}{2}, \pi\}$-turn-restricted cycle cover corresponds to weak rectilinearity of a cycle cover, the stricter $\{\frac{\pi}{2}\}$-turn-restriction describes a cycle covers that is forced to make a right-angle bend at every vertex.

The general cycle cover problem is described formally as follows:

CYCLE COVER (2-FACTOR)
INSTANCE: Graph $G = (V, E)$.
QUESTION: Does there exist a subgraph $H = (V, E')$ such that $E' \subseteq E$ and every $v \in V$ is met by exactly two edges in E'?

The connected cycle cover problem (or Hamiltonian Cycle) is described formally as follows:

CONNECTED CYCLE COVER (HC)
INSTANCE: Graph $G = (V, E)$.
QUESTION: Does there exist a connected subgraph $H = (V, E')$ such that $E' \subseteq E$, every $v \in V$ is met by exactly two edges in E'?

The general cycle cover problem can be reduced to a perfect matching (or 1-factor) problem [LP86] which is solvable in polynomial time [GJ79, Edm65, LP86]. In [LP86], Lovász and Plummer give a good survey of the more general f-factor problem and other related matching problems. It is also well known that the unrestricted connected cycle cover problem is NP-complete [GJ79]. We compare the hardness of CY-

*durocher@cs.ubc.ca
†kirk@cs.ubc.ca

	Cycle Cover		Connected Cycle Cover	
Standard Rectilinear	$\Psi = \{\frac{\pi}{2}\}$	$\Psi = \{\pi, \frac{\pi}{2}\}$	$\Psi = \{\frac{\pi}{2}\}$	$\Psi = \{\pi, \frac{\pi}{2}\}$
crossing	P [O'R88]	P [LP86]	P [O'R88]	NPC [IPS82]
non-crossing	P [O'R88]	NPC [JW93]	P [O'R88]	NPC [Rap86]
Weakly Rectilinear	$\Psi = \{\frac{\pi}{2}\}$	$\Psi = \{\pi, \frac{\pi}{2}\}$	$\Psi = \{\frac{\pi}{2}\}$	$\Psi = \{\pi, \frac{\pi}{2}\}$
crossing	NPC $\bigstar_1$	NPC $\bigstar_2$	P [FW97]	NPC [FW97]
non-crossing	NPC $\bigstar_3$	NPC $\bigstar_4$	P [O'R88, FW97]	NPC [Rap86]

Table 1: Overview of Complexity Results for Ψ-Turn-Restricted Cycle Covers

CLE COVER and CONNECTED CYCLE COVER under standard and weak rectilinearity, under crossing or non-crossing restrictions, and under Ψ-turn-restrictions where $\Psi = \{\frac{\pi}{2}, \pi\}$ or $\Psi' = \{\frac{\pi}{2}\}$.

2 Related Work

O'Rourke [O'R88] shows that a set of vertices in the plane uniquely determines a standard rectilinear polygonal shape; this shape can be determined in polynomial time. Each vertex must be met by two edges, one of which is horizontal, one of which is vertical. The uniqueness of a solution implies the cycle cover and connected cycle cover problems can be solved in polynomial time, under standard rectilinearity and turn-angle restriction $\Psi = \{\frac{\pi}{2}\}$, both for crossing or non-crossing restrictions. See Table 1 for an overview of complexity results.

The assumption that vertices represent a corner can be relaxed to allow edges to continue straight through a vertex. Thus, turn angles must be drawn from $\Psi = \{\pi, \frac{\pi}{2}\}$. Under $\{\pi, \frac{\pi}{2}\}$-turn-restriction, standard rectilinear connected cycle cover is NP-complete [IPS82]. The problem remains hard under the non-crossing restriction [Rap86]. Standard rectilinear cycle cover can be solved by a basic 2-factor algorithm running in polynomial time [LP86]. Under the non-crossing restriction, however, the problem becomes hard [JW93].

Fekete and Woeginger [FW97] show that finding a weakly-rectilinear connected cycle cover is solvable in polynomial time when $\Psi = \{\frac{\pi}{2}\}$ but becomes NP-hard when $\Psi = \{\pi, \frac{\pi}{2}\}$. Weak rectilinearity in a connected graph implies rectilinearity. Furthermore, any rectilinear cycle-cover problem can be reduced to $\binom{n}{2}$ standard rectilinear cycle-cover problem instances, each of which can be checked for a solution using O'Rourke's algorithm [O'R88]. Thus, when $\Psi = \{\frac{\pi}{2}\}$, weakly-rectilinear non-crossing connected cycle cover can be solved in polynomial time. Finally, it is possible to transform any geometric graph G into a geometric graph G' such that G has a non-crossing connected rectilinear cycle cover if and only if G' has a non-crossing connected standard rectilinear cycle cover. In this way, Rappaport's result [Rap86] can be extended to show that when $\Psi = \{\pi, \frac{\pi}{2}\}$, weakly-rectilinear non-crossing connected cycle cover remains NP-hard.

Kratchovíl *et al.* examine the generalized problem of non-crossing restrictions on topological graphs. Given a topological graph G and a property P, does there exist a subgraph $H \subseteq G$ with property P such that the edge set of H is non-crossing? The problem is more interesting when determining whether G has a subgraph H with property P is solvable in polynomial time, but requiring that H be non-crossing renders the problem hard. They demonstrate NP-hardness for several properties P, including the existence of a non-crossing cycle and the existence of a non-crossing k-factor for $k = 1, 2, 3, 4, 5$ (if multi-edges are disallowed, a non-crossing k-factor is impossible for $k \geq 6$).

Formann and Woeginger [FW90], generalize O'Rourke's problem to $k \geq 3$ fixed side orientations. They show given a set of points P, and a set of orientations $D \subseteq [0, \frac{\pi}{2})$, $|D| \geq 3$, determining whether there exists a polygon whose vertices are P and whose edges all have orientation in D is NP-hard.

3 Overview of Proofs

All cycle cover problems discussed here can easily be shown to be in NP. It remains to establish they are NP-hard. Our proofs are by reduction from the following common NP-complete problem:

4-DIMENSIONAL MATCHING (4DM)
INSTANCE: Four sets of elements W, X, Y, and Z, and one set of quadruples $T = \{(w_i, x_j, y_k, z_l)\} \subseteq W \times X \times Y \times Z$.
QUESTION: Does there exist a subset $T' \subseteq T$ such that every $w \in W$, $x \in X$, $y \in Y$, and $z \in Z$ appears exactly once in T'?

Garey and Johnson write, 3-DIMENSIONAL MATCHING "also remains NP-complete if no element occurs in more than three triples, but is solvable in polynomial time if no element occurs in more than two triples." [GJ79, page 221] By a simple reduction, the analogous proposition holds for quadruples within 4DM.

The following are overviews of the components used to reduce 4DM to each of four variations of the orthogonal cycle cover problem.

$\bigstar_1$ WEAKLY-RECTILINEAR $\{\frac{\pi}{2}\}$-TURN-RESTRICTED CYCLE COVER[1]
INSTANCE: Geometric graph $\hat{G} = (V, E, \delta)$.

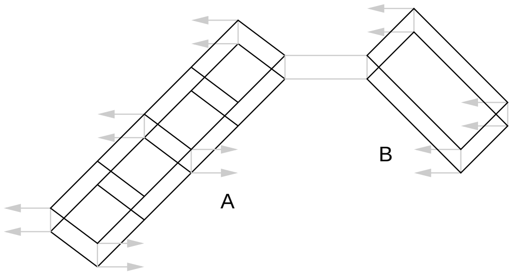

Figure 1: element and quadruple components

QUESTION: Does there exist a cycle cover of $\hat{G}$ such that every vertex is met by exactly two edges with turn angle $\theta = \frac{\pi}{2}$?

Unlike O'Rourke's ORTHOGONAL CONNECT-THE-DOTS (equivalent to STANDARD RECTILINEAR $\{\frac{\pi}{2}\}$-TURN-RESTRICTED CYCLE COVER) which is solvable in polynomial time, $\bigstar_1$ WEAKLY-RECTILINEAR $\{\frac{\pi}{2}\}$-TURN-RESTRICTED CYCLE COVER is NP-hard. This is easily shown by reduction from 4DM. Figure 1A displays the element component and Figure 1B displays the quadruple component.

$\bigstar_2$ WEAKLY-RECTILINEAR $\{\pi, \frac{\pi}{2}\}$-TURN-RESTRICTED CYCLE COVER
INSTANCE: Geometric graph $\hat{G} = (V, E, \delta)$.
QUESTION: Does there exist a cycle cover of $\hat{G}$ such that every vertex is met by exactly two edges with turn angle $\theta \in \{\pi, \frac{\pi}{2}\}$?

Unlike the last problem, edges of the cycle cover are allowed to pass straight through a vertex as well as turn at right angles. The problem remains NP-hard. The reduction is again from 4DM. First, an edge-bending gadget is required (see Figure 2A). Within any cycle cover, either both or neither of edges e_1 and e_2 may be used. Thus, this gadget allows edges to be "bent". The triangular vertices in Figure 2B represent such bending gadgets. Any instance of 4DM is reduced as follows.

[1]The problems are annotated with symbols $\bigstar_1$ through $\bigstar_4$ for quick reference with Table 1.

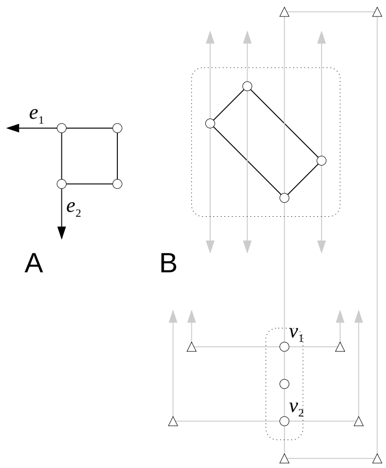

Figure 2: edge-bending gadget

Quadruples consist of a black diagonal box (see Figure 2B). Elements consist of three vertices that align horizontally. An element is linked to the quadruples via a tour of grey edges whose corners consist of triangular vertices. Each tour meets the element at vertices v_1 and v_2.

Each of the last two problems can be further restricted by disallowing edge-crossings. Thus, if two edges cross, at most one of the two may be included in a cycle cover solution.

$\bigstar_3$ NON-CROSSING WEAKLY-RECTILINEAR $\{\frac{\pi}{2}\}$-TURN-RESTRICTED CYCLE COVER
INSTANCE: Geometric graph $\hat{G} = (V, E, \delta)$.
QUESTION: Does there exist a non-crossing cycle cover of $\hat{G}$ such that every vertex is met by exactly two edges with turn angle $\theta = \frac{\pi}{2}$?

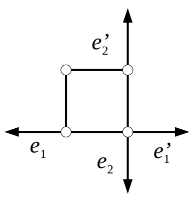

Figure 3: crossing gadget

$\bigstar_3$ NON-CROSSING WEAKLY-RECTILINEAR $\{\frac{\pi}{2}\}$-TURN-RESTRICTED CYCLE COVER can be shown to be NP-hard by a simple modification to the reduction used for $\bigstar_1$ WEAKLY-RECTILINEAR $\{\frac{\pi}{2}\}$-TURN-RESTRICTED CYCLE COVER. Whenever two edges cross at a right angle, their crossing can be replaced by the gadget in Figure 3. Given any $\bigstar_1$ WEAKLY-RECTILINEAR $\{\frac{\pi}{2}\}$-TURN-RESTRICTED CYCLE COVER H, this gadget has the property that $e_1 \in H \Leftrightarrow e_1' \in H$ and $e_2 \in H \Leftrightarrow e_2' \in H$. Crossings that do not occur at a right angle are only a concern if both edges might be included in H. The only such occurrence would be when a horizontal edge e between a quadruple q and an element e_1 crosses another element e_2. Since diagonal edges in an element occur in fours, edge e can be made to

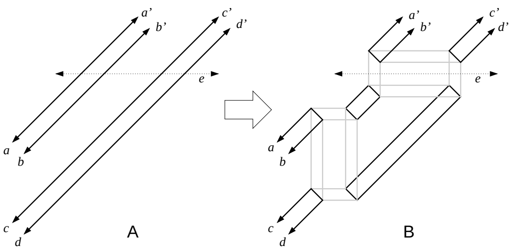

Figure 4: ensuring that edge e only crosses at right angles

cross only vertical edges by replacing with the gadget in Figure 4. This gadget has the property that $\{a, b, c, d\} \subset H \Leftrightarrow \{a', b', c', d'\} \subset H$. Thus, all crossings can be eliminated from the reduction, showing that $\bigstar_3$ NON-CROSSING WEAKLY-RECTILINEAR $\{\frac{\pi}{2}\}$-TURN-RESTRICTED CYCLE COVER is NP-hard.

$\bigstar_4$ NON-CROSSING WEAKLY-RECTILINEAR $\{\pi, \frac{\pi}{2}\}$-TURN-RESTRICTED CYCLE COVER
INSTANCE: Geometric graph $\hat{G} = (V, E, \delta)$.
QUESTION: Does there exist a non-crossing cycle cover of $\hat{G}$ such that every vertex is met by exactly two edges with turn angle $\theta \in \{\pi, \frac{\pi}{2}\}$?

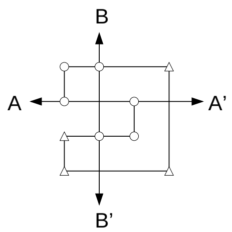

Figure 5: $\{\pi, \frac{\pi}{2}\}$-turn-restricted crossing gadget

Again, we use a modification to the reduction used for $\bigstar_2$ WEAKLY-RECTILINEAR $\{\pi, \frac{\pi}{2}\}$-TURN-RESTRICTED CYCLE COVER. As we did in the last reduction, we replace crossings with a crossing gadget. The gadget in Figure 5 has the property that given any weakly-rectilinear $\{\pi, \frac{\pi}{2}\}$-turn-restricted cycle cover H, $A \in H \Leftrightarrow A' \in H$ and $B \in H \Leftrightarrow B' \in H$ if crossings are disallowed.

It is not difficult to extend these reductions such that each problem ($\bigstar_1$ – $\bigstar_4$) remains NP-hard when the input is given as set of vertex positions (as in [O'R88, Rap86, FW97]) from which all $\binom{n}{2}$ pairs of vertices form an edge in the input graph.

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