An exact algebraic predicate for maintaining the topology of the Additively Weighted Voronoi diagram *

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1 Introduction

The ordinary Voronoi diagram has been extended or generalised in several directions, and these generalised Voronoi diagrams have found many practical applications (see [3] for a general survey on Voronoi diagrams). Abstract Voronoi diagrams [6] include a large number of concrete Voronoi diagrams, e.g., Voronoi diagrams for point sites under any L_p -metric, $1 \leq p \leq \infty$, or under any convex distance function, whose unit circle is algebraic. Furthermore, they comprise weighted Voronoi diagrams and Voronoi diagrams for line segments or circles under the Euclidean metric.

The Additively Weighted Voronoi diagram is related to the Apollonius Tenth problem. The Appolonius Tenth problem is to find a circle Γ tangent to three given circles C_1, C_2, C_3 (see Fig. 1).

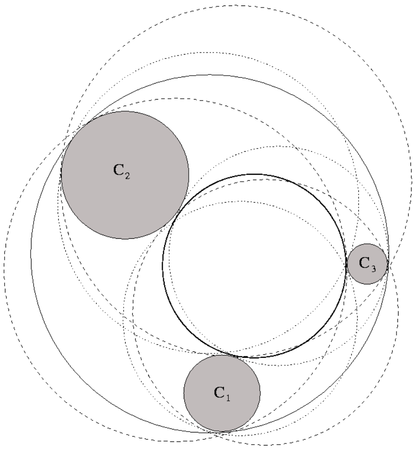

Figure 1: The Apollonius Tenth problem.

In this work, we have developed an exact algebraic predicate of degree 16 for computing the topology of the Additively Weighted Voronoi diagram, i.e. the relationship of adjacency between Voronoi regions of weighted points. This predicate would be the core of an algorithm for constructing incrementally the Additively Weighted Voronoi diagram. This work extends the work of Anton, Mioc and Gold [2] on dynamic Additively Weighted Voronoi diagrams by addressing the robustness issue with an exact predicate. The exact computation of the Additively Weighted Voronoi diagram has not been addressed until Anton et al. [1]. That paper addressed the exact predicate for the offline construction of the dual of the Additively Weighted Voronoi diagram from the dual of the Power Voronoi diagram of spheres by using the relationship between the Additively Weighted Voronoi diagram in the plane and the Power Voronoi diagram of spheres in the three-dimensional space. In this paper, we address a direct exact predicate for maintaining the topology of the Additively Weighted Voronoi diagram in an incremental way. The motivation for an exact predicate lies in the fact that without an exact computation of the topology of the Additively Weighted Voronoi diagram, some geometric and topologic inconsistencies may appear (see Fig. 2,3 and 4). In their independent work, Karavelas and Emiris [5] provided several exact predicates of maximum degree 16 for achieving the same incircle / orientation / edge-conflict-type / difference of radii test as we do in a single predicate. They reduced the degree of their incircle predicate from 28 to 20 and then to 16 using Sturm sequences and invariants.

In the next section we introduce the preliminaries necessary for developing the algebraic predicate. We will explore it in Section 3.

2 Preliminaries

Let $\mathbb{N}$ be the set of integers, $\mathbb{R}$ be the set of reals, and $\mathbb{R}^2$ be the euclidean plane. Let $\mathcal{P} = \{P_1, ..., P_n\}$ be the set of generators or sites, where P_i is the weighted

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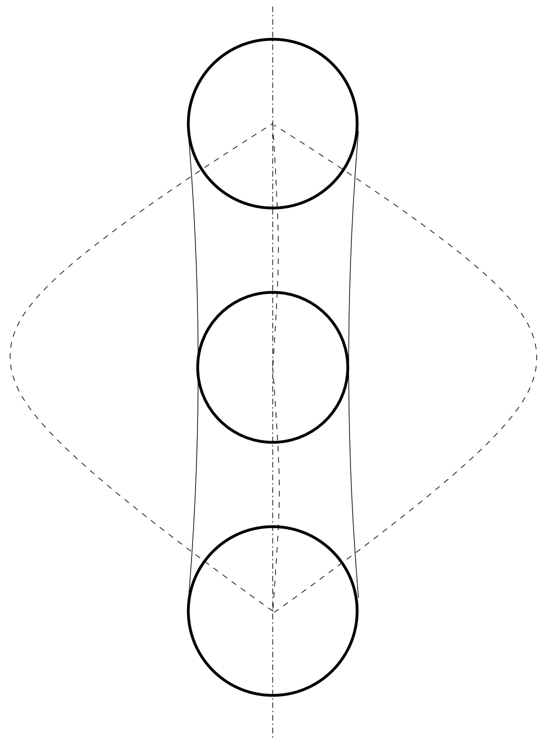

Figure 2: The starting configuration: there are three weighted points (whose corresponding weight circles are drawn in bold) for which the dual of the Voronoi diagram is drawn in dashed lines. All the weighted points are centered on the axis drawn as a dashed and dotted line. Portions of the Appolonius circles tangent to the weight circles have been drawn in plain lines

point located at $p_i \in \mathbb{R}^2$ and of weight $w_i \in \mathbb{R}$. Let C_i be the circle centred at p_i and of radius w_i, which we call weight circle hereafter.

The Voronoi region of P_i is defined by: $\mathcal{V}(P_i) = \{M \in \mathbb{R}^2 | \forall j : d(M, p_i) - w_i \leq d(M, p_j) - w_j\}$, where d is the Euclidean distance between points. The Additively Weighted Voronoi diagram of $\mathcal{P}$ is defined as $\mathcal{V}(\mathcal{P}) = \{\mathcal{V}(P_i) | P_i \in \mathcal{P}\}$ (see Figure 5).

The Additively Weighted Voronoi diagram defines a network composed of edges (loci of points having two nearest neighbours), and vertices (loci of points having three nearest neighbours).

3 The exact algebraic predicate

We consider the maintenance of the topology of the Additively Weighted Voronoi diagram in an incremental way: we check the validity of a triangle of the dual graph of the Additively Weighted Voronoi diagram with respect to a newly inserted weighted point [2]. Thus, four weighted points P_1, P_2, P_3 and P_4 are given: the

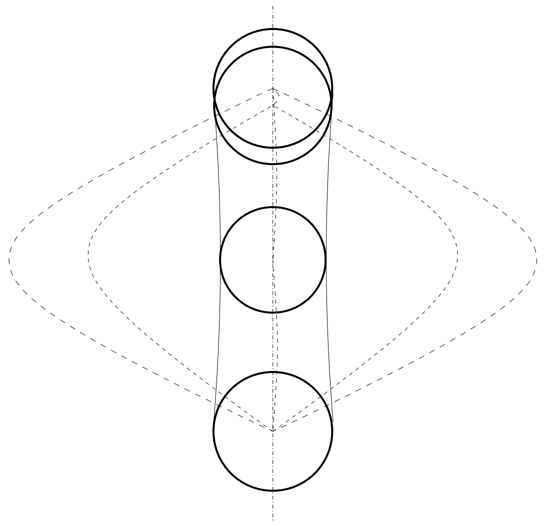

Figure 3: The real configuration after addition of the fourth weigted point (bold weight circle), which is also centered on the axis drawn as a dashed and dotted line.

first three are supposed to define a triangle in the dual graph of the Additively Weighted Voronoi diagram, and the last one is the newly inserted weighted point. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) be the coordinates of p_1, p_2, p_3 and p_4. There are two possible outcomes to the above test of validity: either the triangle is valid with respect to the newly inserted weighted point and the triangle remains in the new dual, or the triangle is not valid with respect to the newly inserted weighted point and the triangle will not be present in the dual any longer. We can see an example of the later case (which we call a switch condition) in Fig. 6. The triangle $P_1P_2P_3$ is not valid with respect to the weighted point P_4, thus the triangles $P_1P_2P_3$ and $P_2P_3P_4$ must switch their common boundary, and become the triangles $P_1P_2P_4$ and $P_1P_3P_4$.

The predicate consists of determining if one of the additive distance from the real vertices among the two additively weighted Voronoi vertices of P_1, P_2 and P_3 to P_4 is greater than or equal to the additive distance to P_1 (see Fig. 6).

The additively weighted Voronoi vertex I with coordinates (x, y) can be obtained algebraically by computing the common intersection of the three circles C'_1, C'_2 and C'_3 (see Fig. 7) expanding (or shrinking) from the three first circles C_1, C_2 and C_3 all at the same rate. The common signed expansion of the first three circles is denoted by r. The circle C'' centred on (x, y) and of radius r is either externally tangent to the first three circles (if the expansion r is positive) or internally tangent to the first three circles (if the expansion r is negative).

The intersection of three expanding circles $I = C'_1 \cap$

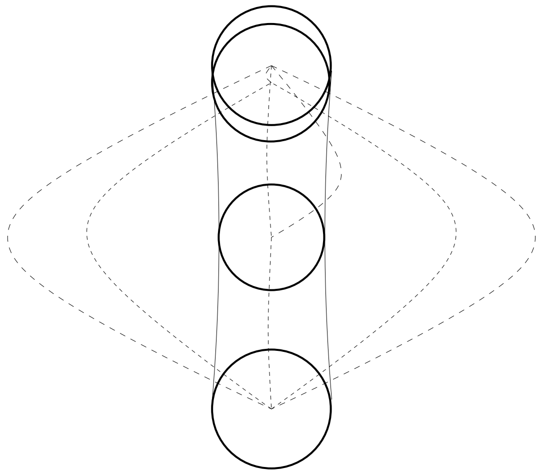

Figure 4: The configuration computed by an approximate algorithm: the difference between real and percieved situations has been exaggerated in order to show the difference. While on the left of the axis of symmetry the weight circle of the newly inserted weighted point has been correctly perceived to intersect the left Appolonius circle, on the right of the axis of symmetry the weight circle of the newly inserted circle has been incorrectly perceived not to intersect the right Appolonius circle. There is an inconsistency within the topology: the crossing of edges in the topology on the right of the axis of symmetry, and an inconsistency between topology and geometry: the topology does not respect the symmetry with respect to the axis of symmetry.

$C'_2 \cap C'_3$ can be computed algebraically as the solution of the following system of three quadratic equations:

$$\begin{cases} (x-x_1)^2 + (y-y_1)^2 - (r_1+r)^2 = 0 \\ (x-x_2)^2 + (y-y_2)^2 - (r_2+r)^2 = 0 \\ (x-x_3)^2 + (y-y_3)^2 - (r_3+r)^2 = 0 \end{cases}$$

Subtracting one of the equations from the remaining two results in a system of 2 linear equations, from which x and y may be expressed in terms of r . Substitution in the first equation then leads to a quadratic equation in r. This means that the unknown quantities x, y, r can be expressed in terms of quadratic radicals as functions of the given centers and radii.

Even though the simplest thing to do at this point would be to compute the two Voronoi vertices and use their computed coordinates and corresponding signed expansion in the computation of the predicate, it is not desirable because the radicals involved in such an intermediate computation would imply more sign determinations as intermediate computations. In order to get the exact predicate in a more elegant and simpler way, we evaluated the predicate without explicitely solving the system of three quadratic equations defining the Voronoi vertex. For this purpose, we used Gröbner

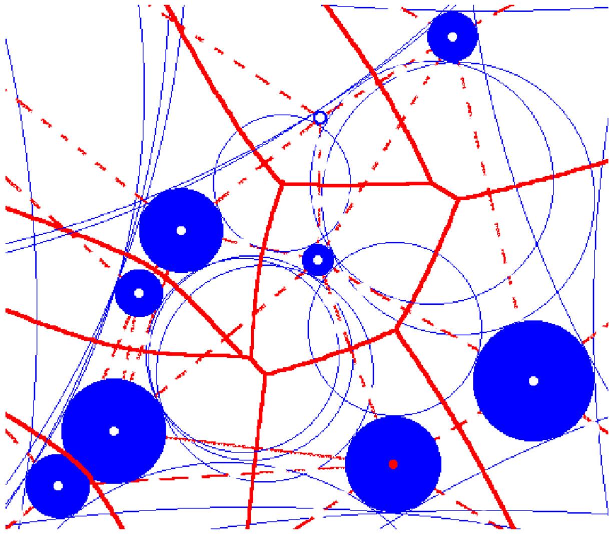

Figure 5: The Additively Weighted Voronoi diagram: the weight circles are drawn as plain disks with a small hole at their centres, the Additively Weighted Voronoi diagram is drawn in plain thick hyperbola segments, and the topology, i.e. the dual is drawn in dashed lines

bases [4]. Gröbner bases are used in Computational Algebraic Geometry in order to compute a canonic representative of the remainder of the division of one polynomial by several polynomials. The Gröbner basis for this system provides a set of polynomials that defines uniquely the algebraic relationships between variables for the solutions of the system. The initial (largest with respect to some monomial order [4]) monomials of each one of the polynomials of the Gröbner basis form an ideal. The monomials that do not pertain to this ideal form a basis for the remainders of the division of a polynomial by the polynomials of the system. The size of this basis corresponds to the number of solutions of the system counted with their multiplicity. In our case, there are two solutions.

We computed the Gröbner basis of the ideal corresponding to $I = C'_1 \cap C'_2 \cap C'_3$. This Gröbner basis consists of the previously mentioned quadratic equation in r and linear equations in x, y and r. We considered the operation of multiplication of polynomials by the polynomial $(x_4-x)^2 + (y_4-y)^2 - (r+r_4)^2$, which expresses the relative position of C_4 with respect to C''. This is a linear mapping. The operation of this mapping on the canonic representative of the remainder of the division of a polynomial by the three polynomials of the system is also a linear mapping that can be expressed by a matrix.

Firstly, we compute the matrix $M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}$ of the following mapping $M_{(x_4-x)^2+(y_4-y)^2-(r+r_4)^2}$:

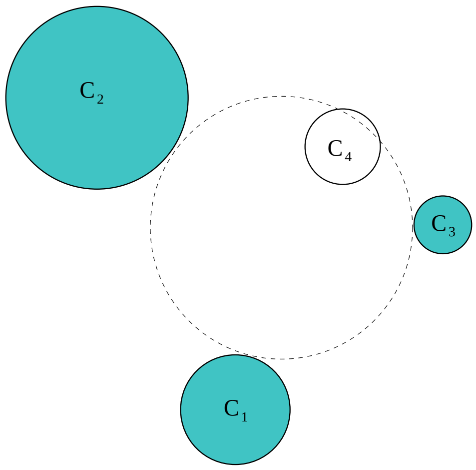

Figure 6: The predicate for maintaining the topology of the Additively Weighted Voronoi diagram

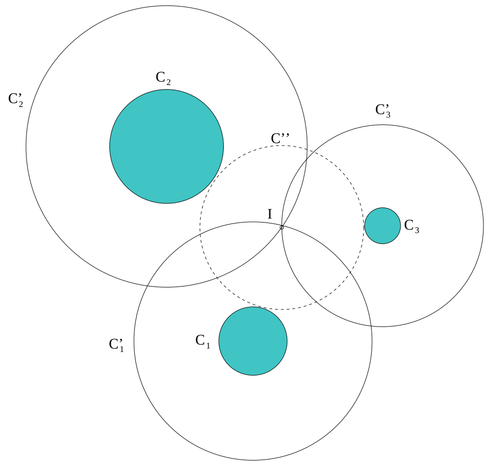

Figure 7: The Additively Weighted Voronoi vertex as the common intersection of three expanding/shrinking circles

$[p] \longrightarrow \left[\left((x_4 - x)^2 + (y_4 - y)^2 - (r + r_4)^2\right) p\right].$

The eigenvalues of this matrix are the values of $(x_4 - x)^2 + (y_4 - y)^2 - (r + r_4)^2$ taken by the possible values of the solutions (x, y, r) of the system (see Theorem 4.5, page 54 in [4]). The eigenvalues of M are the solutions of $det\,(M - \lambda Id) = 0$, i.e. the roots of

$$\lambda^2 - \lambda\,(m_{00} + m_{11}) + (m_{00}m_{11}) - (m_{01}m_{10}) = 0 \quad (1)$$

The values of the predicate are determined by the sign of the roots of Equation 1 (which are the eigenvalues of M). If there is only one eigenvalue and it is 0 then the fourth circle is tangent to the circle externally tangent to the first three circles. If the $sgn(\Delta)$ (where $\Delta = (m_{00} + m_{11})^2 - 4\,(m_{00}m_{01} - m_{01}m_{10})$) is negative then there is no switch condition (see Fig. 8). If the $sgn(\Delta)$ is 0 or positive then we have to evaluate the sign of the roots of the quadratic equation.

When there is only one double root of Equation 1 then we have the following two possibilities: either the value of the root of Equation 1 is positive or 0 and the triangle will not switch, or the value of the root of Equation 1 is negative and the triangle will switch (see Fig. 6). When there are two real roots of Equation 1 we have two triangles to consider (see Fig. 9). The triangles that belong to the roots with a negative value will switch (see Fig. 9).

The total degree of this algebraic predicate (i.e. of the polynomial whose sign is the value of the predicate) is 16. There is not enough space to show the elements of the matrix of the multiplication operator here, but the Macaulay 2 code is presented in the appendix.

4 Conclusions

We have provided an exact predicate for the computation of Voronoi diagrams for circles in the plane. The degree of this predicate is 16.

5 Acknowledgments

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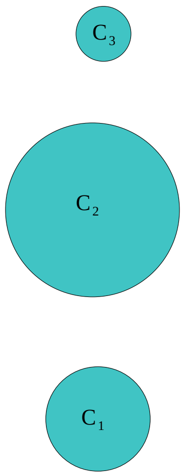

Figure 8: No switch due to the absence of real Voronoi vertex

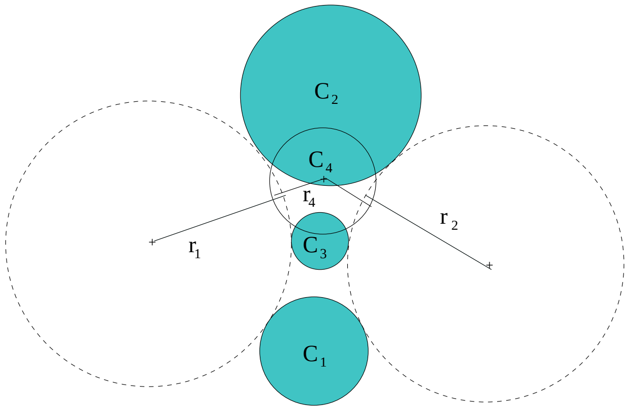

Figure 9: Two triangles can be possibly switched simultaneously by the addition of a single weighted point

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APPENDIX

Macaulay 2 code for the predicate:

```
gbTrace 4
dim FractionField := F -> 0
P = frac(QQ[a, b, c, d, e, f, g, h, i, j, k, l])
R = P[x, y, t]
cercle1 = (x - a)^2 + (y - b)^2 - (c + t)^2
cercle2 = (x - d)^2 + (y - e)^2 - (f + t)^2
cercle3 = (x - g)^2 + (y - h)^2 - (i + t)^2
emptycircle = ideal(cercle1, cercle2, cercle3)
ecgb = gb emptycircle
print ecgb
eckb = basis cokernel gens ecgb
print eckb
kl = sort(flatten(entries(eckb)))
kmind = splice {0..#kl - 1}
scan(kl, entry-> print ring entry);
hashlist = pack(2, mingle(kl, kmind));
feetmon = applyKeys(hashTable hashlist, key->
toString(key));
compmat = f -> (htl = apply(kl, be->
hashTable(pack(2, mingle(apply(flatten(entries(
(coefficients((f * be)%ecgb))#0)),
item -> feetmon#(toString(item))), flatten(
entries((coefficients((f * be)%ecgb))#1))))));
matrix(table(#kl, #kl, (i, j)-> if (htl#i)#?j then
(htl#i)#j else 0)));
matp2 = compmat((x - j)^2 + (y - k)^2 - (t + l)^2);
m00 = matp2_(0, 0)
m01 = matp2_(0, 1)
m10 = matp2_(1, 0)
m11 = matp2_(1, 1)
cm00 = coefficients m00
cm000 = cm00#0
cm001 = cm00#1
cm01 = coefficients m01
cm010 = cm01#0
cm011 = cm01#1
cm10 = coefficients m10
cm100 = cm10#0
cm101 = cm10#1
cm11 = coefficients m11
cm110 = cm11#0
cm111 = cm11#1
```