The complexity of flow diagrams in the plane*

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Abstract

We derive tight bounds on the combinatorial and algorithmic complexity of flow diagrams induced by weighted points in the plane. Flow diagrams resemble power diagrams, but have different complexities.

1 Introduction

A smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ gives rise to a smooth vectorfield on $\mathbb{R}^2$, namely the gradient vectorfield that maps every point x of $\mathbb{R}^2$ to the gradient of f at x. The gradient vector field in turn gives rise to an ordinary differential equation

$$\frac{d}{dt}\phi(t,x) \;=\; \mathsf{grad}\, f(\phi(t,x)).$$

The theory of ordinary differential equations states that this equation is integrable. The solution of the equation is a mapping

$$\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2,$$

which has two remarkable properties, first $\phi(0,x) = x$ and second $\phi(t,\phi(s,x)) = \phi(t+s,x)$. The first parameter of ϕ can be interpreted as time and the mapping itself tells how the points of $\mathbb{R}^2$ move in time. The two conditions state that, first the points have not moved at time zero and second any point x moves in time $t+s$ to the point where the point $\phi(s,x)$ moves in time s. The function ϕ is called a *dynamical system* or *flow* on $\mathbb{R}^2$. A point x with $\phi(x,t) = x$ for all $t \in \mathbb{R}$ is called a fixpoint of the dynamical system. The fixpoints of ϕ are exactly the critical points of f. Fixing the second parameter of ϕ provides for every $x \in \mathbb{R}^2$ a mapping $\phi_x : \mathbb{R} \to \mathbb{R}^2, t \mapsto \phi(t,x)$, which is called the orbit of x. The orbit describes the motion of the point x in time. This motion always follows the direction of steepest ascent of the function f, i.e. the orbits are integral curves to the gradient vector field of f.

In this paper we study a distance function h defined via a set of weighted points in the plane. It turns out that this function is not smooth. It is only smooth in the interiors of the power cells of the disks. Thus we cannot apply the theory of ordinary differential equations to get a dynamical system from h. But as in the case of smooth functions there is a unique direction of steepest ascent of h at every non-critical point of h. Assigning to the critical points to h the zero vector and to every other point of $\mathbb{R}^2$ the unique unit vector of steepest ascent gives rise to a vector field on $\mathbb{R}^2$. Note that this vector field is not smooth. We are going to construct a mapping

$$\phi : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2,$$

such that at every point $(t,x) \in [0,\infty) \times \mathbb{R}^2$ the right derivative

$$\lim_{t \leftarrow t'} \frac{\phi(t,x) - \phi(t',x)}{t - t'}$$

exists and is equal to the unique unit vector of steepest ascent at x. The mapping ϕ has the two properties of a dynamical system, but it is defined only for non-negative times. The latter is due to the non-smoothness of h. We call ϕ the flow induced by the weighted points. The fixpoints of the flow ϕ are exactly the critical points of the distance function h. We call the set of all points in $\mathbb{R}^2$ that flow into a maximum m of h the max-region of m. The max-regions are polygons. In the unweighted case the boundaries of the max-regions are given by the Gabriel graph of the point set. The Gabriel graph contains $\Theta(n)$ vertices, edges and regions and can be computed in time $\Theta(n \log n)$. Here we show that in the weighted case the boundary graph is still connected but contains $\Theta(n^2)$ vertices and edges and can be computed in time $\Theta(n^2)$.

The idea to study the flow induced by weighted points goes back to [1], where a discrete approximation of ϕ was studied. We successfully applied the three-dimensional version of the dynamical system ϕ to surface reconstruction [3] and to compute pockets in proteins [4].

2 Power height function

Weighted point. A weighted point is a pair $(z,r) \in \mathbb{R}^3$, where $z \in \mathbb{R}^2$ denotes the point and r denotes its weight. We refer to r also as the power of the point.

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Power distance. The power distance of a point $x \in \mathbb{R}^2$ from a weighted point (z, r) is $\pi(x) = \|x - z\|^2 - r^2$.

Power height function. Let B be a set of weighted points in $\mathbb{R}^2$. The power height function is given as

$$h(x) = \min\{\pi_i(x) \,:\, b_i \in B\}.$$

Observe that the function h is continuous. It is smooth everywhere besides at points which have the same power distance from two or more weighted points, i.e. at points that lie on the boundary of power cells.

Critical- and regular points. The critical points of the power height function h are the intersection points of Voronoi objects V and their dual Delaunay object σ. The index of a critical point is the dimension of σ. All points which are not critical are called regular.

3 Induced flow

We want to associate to every power height function a dynamical system whose fixpoints are exactly the critical points of this function.

Driver. Given $x \in \mathbb{R}^2$. Let V be the lowest dimensional power object in the power diagram of B that contains x and let σ be the dual Delaunay object of V. The driver of x is the point on σ closest to x.

Induced flow. The flow ϕ induced by B is given as follows: For all critical points x of the height function associated with B we set:

$$\phi(t, x) = x \;,\; t \in [0, \infty)$$

Otherwise let y be the driver of x and R be the ray originating at x and shooting in the direction $x - y$. Let z be the first point on R whose driver is different from y. Note that such a z need not exist in $\mathbb{R}^2$. In this case let z be the point at infinity. We set:

$$\phi(t, x) = x + t\frac{x - y}{\|x - y\|} \;,\; t \in [0, \|z - x\|]$$

For $t > \|z - x\|$ the flow is given as follows:

$$\begin{aligned} \phi(t, x) &= \phi\left(t - \|z - x\| + \|z - x\|,\, x\right) \\ &= \phi\left(t - \|z - x\|,\, \phi\left(\|z - x\|,\, x\right)\right) \end{aligned}$$

Here it is not completely obvious, but ϕ can be shown to be well defined on the whole of $[0, \infty) \times \mathbb{R}^2$. Furthermore, the following two properties of dynamical systems hold for induced flows,

(1) $\phi(0, x) = x$.

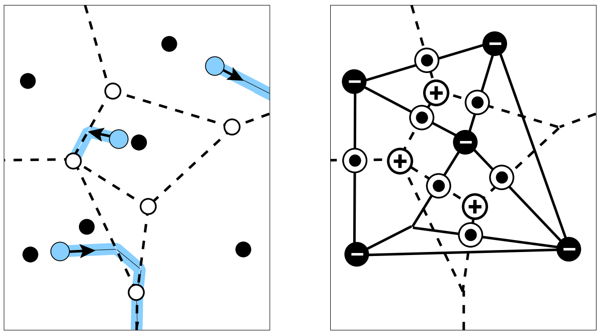

Figure 1: Some orbits of the flow induced by some weighted points (on the left). The minima $\ominus$, saddles $\odot$ and maxima $\oplus$ of the corresponding distance function (on the right). Note that the pictures also show the power- and Delaunay diagram of the weighted point set.

(2) $\phi(t, \phi(s, x)) = \phi(t + s, x)$.

Stable manifolds. Given an advancing dynamical system ϕ on a manifold $\mathbb{R}^2$. The stable manifold $S(x)$ of a fixpoint $x \in \mathbb{R}^2$ contains all points that flow into x, i.e.

$$S(x) \;=\; \{y \in \mathbb{R}^2 \;:\; \lim_{t \to \infty} \phi_y(t) = x\}.$$

The central theme of our discussion will be the stable manifolds of saddles.

Lemma 1 [2] *Let s be a saddle of ϕ. If the stable manifold $S(s)$ of s does not contain a power vertex then* closure$(S(s))$ *is a piecewise linear curve.* $\square$

In the following we want to assume the stable manifolds of all the saddles are piecewise linear curves.

Stable decomposition graph. Let G be the following graph: Its vertex set consists of all minima and saddles. Two vertices are connected by an edge if there exists a stable manifold $S(s)$ of some saddle s such that the two vertices are both contained in closure$(S(s))$ and there is no other vertex on $S(s)$ between them. We have to take the closure of the stable manifolds because otherwise the minima would not be contained in them. We refer to G as the stable decomposition graph associated with an induced flow. Let S be the set of saddles of ϕ and $T(S) = \bigcup_{s \in S} \text{closure}(T(s))$. We refer to $T(S)$ as the geometric realization of G. This realization defines the flow diagram, i.e. the vertices, edges and regions of the graph.

Instead of working directly with stable manifolds of maxima we introduce max-regions which have nicer properties.

Max-regions. Given an induced flow. If m is a maximum of this flow we call the closure of the stable manifold $S(m)$ the Max region of m.

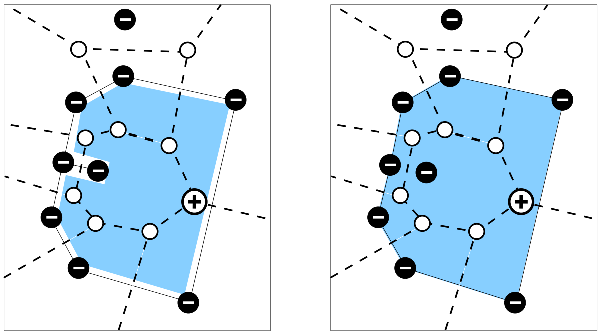

Figure 2: A stable manifold of a maximum $\oplus$ (on the left) and its corresponding Max region (on the right). Note that the saddles are not marked in these figures.

Theorem 1 [2] *Let ϕ be an induced flow. The* max-*regions of ϕ are exactly the closures of the bounded regions of the geometric realization of the stable decomposition graph associated with ϕ.* □

Corollary 1 [2] *The stable decomposition graph associated with ϕ is connected.* □

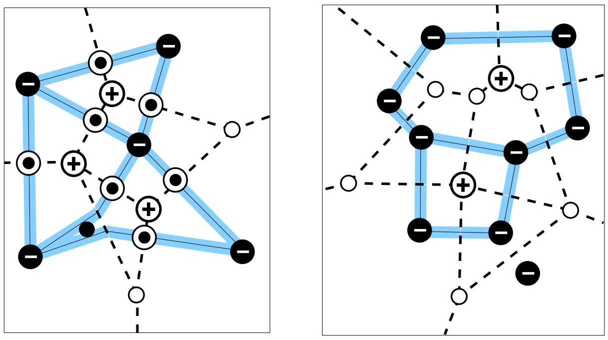

Figure 3: The flow diagram of the set of weighted points from Figure 1 (on the left) and another flow diagram (on the right).

4 Combinatorial complexity

Lemma 2 *The geometric realization of the stable decomposition graph of a flow induced by a set B of n weighted points contains $O(n^2)$ straight line segments.*

PROOF. The power diagram of B has at most $2n-5$ vertices and at most $3n-6$ edges. That is, the power diagram of B contains $O(n)$ power objects. It can be shown that every point in the interior of some power object has the same driver. This implies that the flow induced by B can have only $O(n)$ different drivers.
The geometric realization of the stable decomposition graph associated with B is the union of the stable manifolds of the saddle points of the flow induced by B. Given such a saddle s, the stable manifold $S(s)$ of s can be decomposed in arcs which have s as one of their endpoints. Thus there are two arcs which have s as one of their endpoint. Every arc can be directed according to the direction of the flow along the arc and it contains the same number of straight line segments as the orbit of some point very close to its beginning. Since every driver controls the flow of this point at most once its orbit consists of $O(n)$ straight line segments. The number of saddles is at most the number of power edges, i.e. there $O(n)$ saddles. Since for every saddle there are only two arcs this implies that the geometric realization of the stable decomposition graph associated with B contains $O(n^2)$ straight line segments. □

Next we present a construction that shows that the upper bound from Lemma 2 can be matched. That is, the maximum complexity of the stable decomposition graph of a set of n weighted points in terms of the number of straight line segments is $\Theta(n^2)$.

Lemma 3 *The maximum number of straight line segments in the geometric realization of the stable decomposition graph of n weighted points is $\Omega(n^2)$.*

Our lower bound construction for sufficiently large $n \in \mathbb{N}$ goes as follows: Let k be $\lfloor n/2 \rfloor$ if this expression is odd and $\lfloor n/2 \rfloor + 1$ otherwise, i.e. k is always odd. Let $m = n - k$. We place m weighted points $b_1, \ldots, b_m$ on a vertical line L_1 such that the indices of the points represent the ordering of the points along L_1. We choose some power for the point b_m and adapt the power of b_{m-1} such that the power bisector of b_m and b_{m+1} lies above b_m. We set the power of b_i for $i = m-2, \ldots, 1$ such that the power bisector of b_i and b_{i+1} lies below the power bisector of b_{i+1} and b_{i+2} but above b_m. The k remaining points $c_1, \ldots, c_k$ are placed on a line L_2 orthogonal to L_1 that intersects L_1 above b_m. Let d_1 denote the distance between b_m and L_2. The indices of these points represent the ordering the points along L_2. The point $c_{\lceil k/2 \rceil}$ is placed on the intersection of L_1 and L_2. We choose the power of $c_{\lceil k/2 \rceil}$ such that the power bisector of $c_{\lceil k/2 \rceil}$ and b_m intersects L_1 in distance d_2 below L_2. We keep the distance d_2 fixed in contrast to d_1 which has only to be large enough to guarantee that the power bisector of $c_{\lceil k/2 \rceil}$ and b_m lies above the power bisector of b_{m-1} and b_m. We assign the same power to all $c_i, i = 1, \ldots, k$ as we assigned to $c_{\lceil k/2 \rceil}$. Note that this power is a function of d_1.

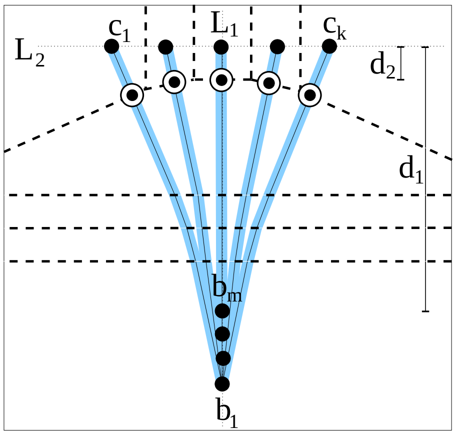

Figure 4: Nine (weighted) points •, the corresponding power diagram and the stable manifolds of the five saddles ⊙.

By construction the following pairs of points give rise to a power edge: (b_i, b_{i+1}) for $i = 1, \ldots, m-1$, (c_1, b_i) for $i = 1, \ldots, m-1$, (c_i, b_m) for $i = 1, \ldots, k$, (c_k, b_i) for $i = 1, \ldots, m-1$ and (c_i, c_{i+1}) for $i = 1, \ldots, k-1$. We can bring the smaller angle made by L_1 and all power edges besides the edges corresponding to the pairs (c_i, c_{i+1}) arbitrarily close to $\pi/2$ by increasing d_1 while keeping d_2 fixed, i.e. by moving L_2 further away from b_m. At the same time the angle made by L_2 and the Delaunay edges dual to the power edges corresponding to the pairs $(c_i, b_m), i = 1, \ldots, k$ comes arbitrarily close to $\pi/2$. Since the distance d_2 remains fixed this implies that these Delaunay edges have to intersect their dual power edges, i.e. we have k saddles corresponding to the pairs $(c_i, b_m), i = 1, \ldots, k$.
The stable manifold of the saddle s that corresponds to the pair $(c_i, b_m), i \in \{1, \ldots, k\}$ has non empty intersection with the interior of the Min region that corresponds to b_1. That is, part of the stable manifold of s connects b_1 with s without containing b_1. To do so this part of the stable manifold has to cross all power edges corresponding to pairs $(b_i, b_{i+1}), i = 1, \ldots, m-1$. That is, the stable manifold of s contains at least m straight line segments. By definition stable manifolds of different saddles cannot meet. Thus the geometric realization of the stable decomposition graph contains at least $km = \Theta(n^2)$ straight line segments. □

Theorem 2 *The maximum complexity of the geometric realization of the stable decomposition graph of n weighted points is $\Theta(n^2)$ vertices, $\Theta(n^2)$ straight line segments and $\Theta(n)$ regions.*

Proof. From Theorem 1 we know that the geometric realization of the stable decomposition graph decomposes the plane into $O(n)$ regions, because there cannot be more regions than power vertices. This upper bound is matched by the number of Max regions in the lower bound construction in Lemma 3 for the number of line segments. This in turn implies together with the Euler formula for planar graphs and Lemma 2 that the geometric realization of the stable decomposition graph has $O(n^2)$ vertices. This upper bound is again matched by the number of vertices in the lower bound construction in Lemma 3. That is, the maximum complexity of the geometric realization of the stable decomposition graph is $\Theta(n^2)$ vertices, $\Theta(n^2)$ straight line segments and $\Theta(n)$ regions. □

Corollary 2 *The geometric realization of the stable decomposition graph of a set of n weighted points has a worst case algorithmic complexity of $\Theta(n^2)$.*

Proof. Every algorithm to compute the stable decomposition graph has to look at each vertex and each edge of the graph at least once. Thus the lower bound construction in Lemma 3 for the maximum geometric complexity of stable decomposition graphs provides also a lower bound of $\Omega(n^2)$ for the worst case algorithmic complexity of geometric realizations of stable decomposition graphs. This lower bound is matched by the worst case running time of the algorithm that we presented in [2]. Hence the worst case algorithmic complexity of the geometric realizations of stable decomposition graphs is $\Theta(n^2)$. □

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