A Near-Quadratic Algorithm for the Alpha-Connected Two-Center Decision Problem

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Abstract
We consider the problem of deciding if a set of points can be covered by two discs with centers p and q and common radius r such that the ratio $d(p,q)/r$ is bounded below by a user supplied constant α. We also present an $O(n^2 log^2 n)$ algorithm for this problem.

Keywords: Computational geometry, two-center decision problem, alpha-connected two-center decision problem,Center Hull

1 Introduction

This paper considers a variant of the two-center decision problem [6], alpha-connected two-center decision problem (abbreviate to α-C2CD problem). Given a set of points S in the plane, the radius r and the constant α, the α-C2CD problem is to determine whether S can be covered by two congruent closed discs with centers p and q and common radius r such that the ratio $d(p,q)/r$ is bounded below by the constant α. One of the applications of the α-C2CD problem is to set up the medical centers. Think of the problem where we want to place two emergency medical units in the area of n given sites such that the worst-case response distance to any site is bounded below r. If these two units are wholly independent, the two-center problem, then everything will be okay. However, realistically, it is possible that one unit may request blood or technical support from the other unit for emergency surgical operations. As a result, to limit the distance of these two units seems somewhat necessary.

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The problem proposed by Huang [7] is a special case of the alpha-connected 2-center problem. In that case, each center is just on the boundary of the disc centered at the other center. Huang had given an algorithm with running time $O(n^5)$ for the problem, and it can also be applied, in the same time, to the α-C2CD problem. The main result of this paper is an algorithm which solves the α-C2CD problem in $O(n^2 log^2 n)$ time. The algorithm is conceptually simple and has much explicit geometric flavor.

2 The constrained alpha-connected two-center decision problem

In this section, we will introduce the *constrained α-C2CD problem*, which is the base-problem for our algorithm to solve the α-C2CD problem.

Let $S = \{s_1, s_2, \ldots, s_n\}$ be a set of points in the plane. The Euclidean distance between two points p and q in the plane is denoted by $d(p,q)$. Given a point p in the plane and a radius r, we denote the circle with center p and radius r by $C(p,r)$. We say that $C(p,r)$ *covers* S if for $i = 1, ..., n$, $d(p, s_i) \leq r$. A pair of circles $C(p,r)$, $C(q,r)$ that intersect each other (i.e., $d(p,q) \leq 2r$) is called *connected two-circle* and is denoted by $C2C(p,q,r)$. We say that $C2C(p,q,r)$ *covers S* if for $i = 1, ..., n$, $d(p,s_i) \leq r$ or $d(q,s_i) \leq r$. The *connection ratio* β of the $C2C(p,q,r)$ is defined as $1 - d(p,q)/2r$, and $\alpha - C2C(p,q,r)$, called *α-connected two-circle*, denotes the two-circle $C2C(p,q,r)$ with the connection ratio β and $\beta \geq \alpha$, i.e., the distance between the two centers, $d(p,q)$, is not great than $2(1-\alpha)r$. Note that if $p = q$, $C2C(p,q,r)$ becomes one circle. $\alpha - C2C(p,q,r)$ is called *strongly connected* if $1/2 \leq \alpha \leq 1$, and *weakly connected* otherwise.

Now we redefine the α-C2CD problem as follows:

Definition 1. Given a set S of n points in the plane, the radius r and the constant α, the *α-connected two-center decision problem* is to determine whether there exists an α-connected two-circle $\alpha - C2C(p,q,r)$ that covers S.

The *linear partition* (S_1, S_2) of S is to partition S into two subsets, S_1 and S_2, with a straight line such that $S_1 \cap S_2 = \phi$ and $S_1 \cup S_2 = S$.

Definition 2. Given a linear partition (S_1, S_2) of S, the radius r and the constant α, the *constrained α-connected two-center decision problem* is to determine whether there exists an $\alpha - C2C(p,q,r)$ such that S_1 and S_2 can be covered by $C(p,r)$ and $C(q,r)$ respectively.

Theorem 1. Given a point set S, the radius r and the constant α, there exists a constrained α-C2CD problem whose solution is the same as the α-C2CD problem.

Proof. We prove it by generating a constrained α-C2CD problem which has the same solution as the original α-C2CD problem. Let α-C2C(p,q,r) be the solution of the α-C2CD problem. We can generate a linear partition (S_1, S_2) by drawing a straight line l which is the perpendicular bisector of the line segment pq. That is, line l passes through the cross points of the two circles $C(p,r)$ and $C(q,r)$. Obviously, S_1 and S_2 can be covered by $C(p,r)$ and $C(q,r)$ respectively. Then by definition, α-C2C(p,q,r) is also the solution of the corresponding constrained α-C2CD problem with linear partition (S_1, S_2). $\square$

The immediate consequence of this theorem is that we can solve the α-C2CD problem by solving all of the possible instances of the constrained α-C2CD problem (each with different linear partition). Since there are totally $O(n^2)$ possible partitions, our algorithm tries to find the answer of the α-C2CD problem from the answers of the $O(n^2)$ constrained α-C2CD base-problems. In the following sections, we will first introduce the notion of center-hull. Then we show that the constrained α-C2CD problem is, in fact, a distance problem between two center-hulls.

3 The center-hull

In this section, we first introduce the notion and properties of center-hull which has been widely used in solving numerous geometric problems and is crucial to the development of our algorithm. The authors in [1, 4] call *center-hull circle-gon* or *circular-hull*, and had it well-defined. For our explanation, we redefine it as follows:

Definition 3. The *center-hull* of S with radius r, denoted $CENH(S,r)$, is the locus of points which are the centers of the circles covering S and with radius r, that is, $CENH(S,r) = \{p | C(p,r) \text{ covers } S\}$.

The following two properties are quite straightforward.

Property 1. Let r_0 be the radius of the smallest enclosing circle of S. $CENH(S,r)$ is empty if $r < r_0$.

Property 2. A point p is in $CENH(S,r)$ if and only if the distance between p and its farthest neighbor in S is less than or equal to r.

Also, center-hull has several important properties.

Property 3. Center-hull is convex.

Property 4. A point p is on $\partial CENH(S,r)$, the boundary of the $CENH(S,r)$, if and only if the distance between p and its farthest neighbor in S is equal to r.

The *farthest-point Voronoi diagram* [8] of s set of points S, denoted by $FPVD(S)$, is defined as follows: the *Voronoi region* $V(s_i)$ associated with an element s_i from S is the set of all points in the plane that are farther, or as far, from s_i than from any other element in S and $FPVD(S)$ is the union of all regions $V(s_i)$, $s_i \in S$.

The following relationships between center-hull and the farthest-point Voronoi diagram of a given point set stem from Property 4.

Property 5. Within each Voronoi region, say $V(s_i)$, $\partial CENH(S,r)$ is a subarc with center s_i and radius r.

Property 6. $\partial CENH(S,r)$ is a circular list of subarcs with common radius r and each centered at the associated farthest-neighbor in S.

We call the center of one subarc on $\partial CENH(S,r)$ the *control point* of that subarc, and then $CENH(S,r)$ can be described with the circular list of the control points together with the radius r.

4 Solving the constrained α-connected two-center decision problem

In this section, we present an algorithm for solving the constrained α-C2CD problem. In fact, center-hull $CENH(S,r)$ is the locus of centers of the circles covering S and with radius r. The key idea of our algorithm is to transform the constrained α-C2CD problem into a distance problem between two center-hulls.

Definition 4. The distance between $CENH(S_1,r)$ and $CENH(S_2,r)$ is defined as $Min\{d(p,q)|p \in CENH(S_1,r), q \in CENH(S_2,r)\}$, and p, q are called the *nearest points of the center-hulls.*

From above definition, it is easy to see that the nearest points of two center-hulls lie on the boundaries of the center-hulls.

Theorem 2. If the distance between $CENH(S_1,r)$ and $CENH(S_2,r)$ is less than or equal to $2(1-\alpha)r$, then there exists an α-connected two-circle α-C2C(p,q,r) that is a solution to the constrained α-C2CD problem with linear partition (S_1,S_2).

Proof. Let p, q be the nearest points of $CENH(S_1,r)$ and $CENH(S_2,r)$, where p in $CENH(S_1,r)$ and q in $CENH(S_2,r)$. From the property of center-hull described at the beginning of this section, we can draw two circles $C(p,r)$ and $C(q,r)$ that cover S_1 and S_2 respectively. Moreover, since $d(p,q) <= 2(1-\alpha)r$, $C(p,r)$ and $C(q,r)$ form a α-connected two-circle α-C2C(p,q,r). Then by the definition of constrained α-C2CD problem, it follows. □

Theorem 2 gives us the idea to solve the constrained α-C2CD problem (with a given linear partition) by solving the distance problem between two center-hulls. The outline of our algorithm to solve the constrained α-C2CD problem is as follows:

Algorithm: The constrained α-C2CD algorithm

Input: A point set S, the linear partition (S_1,S_2), the radius r and the constant α.

Output: the answer to the constrained α-C2CD problem

Step 1 Enumerate all bipartitions in a sequence, where passing from one partition to the next amounts by moving one point from one subset to the other one.

Step 2 Dynamically construct the two farthest-point Voronoi diagrams and further the two center-hulls of the two subsets.

Step 3 Compute the distance, say d, between the two center-hulls.

Step 4 If $d \leq 2(1-\alpha)r$, then return YES, i.e., S can be covered by two congruent closed discs with radius r and the connection ratio of the two centers, β, is at least α. Otherwise, go to Step 2 and repeatedly try the next bipartition until the solution has been found or all of the possible bipartitions have been tried.

Theorem 3. The α-C2CD problem can be solved in $O(n^2 log^2 n)$ time.

Proof. The requirement that the centers lie close to each other makes this variant a subcase of the situation where the centers are near each other, as studied in the papers of Sharir [9], Eppstein [4] and Chan [2]. The intersection area of the two covering circles, $C_1(p,r)$ and $C_2(q,r)$, is at least a certain value, whereas the entire S can be covered by, say, an axis-parallel square R of size $4r$, which we can easily compute in $O(n)$ time. It follows that we can construct $O(1)$ points within R, so that at least one of them will lie in $C_1 \cap C_2$ (and fairly close to both p and q), if a solution exists for the given r. Let o be such a point. We sort the points of S in angular order about o, and partition the sorted list into two subsets, S_1, S_2, by two rays emanating from o. Apply the matrix representation of Sharir [9], where each entry of the matrix represents a partition of S into a left subset S_1 and a right subset S_2. Then we search over all entries of the matrix. When passing from one entry to an adjacent one, one point is moved from S_1 to S_2 or vice versa. We then update the *farthest-point Voronoi diagrams* and center-hulls of S_1 and S_2 as in [4, 5, 9]. We refer the readers to [5] for additional details concerning the dynamic construction of FPVD and center-hulls. And we conclude that Step 2 can be done in $O(log^2 n)$ time. Step 3 computes the distance between two center-hulls, it is nothing but the minimum distance problem as in [3]. The only difference is the algorithm in [3] is for finding the minimum distance of two convex polygons rather two center-hulls. But since center-hull has convexity as convex polygon, the derivation of the algorithm in [3] applies as well in the case of center-hull (by replacing edges of polygons with subarcs of center-hull). Therefore, Step 3 can be done in $O(logn)$ time. Finally, since the initial construction of FPVD and center-hulls can be done in $O(nlogn)$ time and there are totally $O(n^2)$ entries of matrix to be searched, this leads to a $O(n^2log^2n)$ solution. □

5 Conclusions

In this paper, we consider the α-connected two-center decision problem, a variant of the two-center decision problem, and give an algorithm solving this problem in $O(n^2log^2n)$ time. Our algorithm could be improved if the number of bipartitions could be reduced. To achieve this goal, we have tried making use of various techniques such as parametric searching, the use of expanders, matrix searching and data structure for convex programming. These techniques have been effectively used in solving the two-center problem, but we find it seems difficult to apply these techniques in solving the α-connected two-center decision problem.

Our results raise the next question: whether the α-connected two-center problem can also be solved in $O(n^2log^2n)$ time? We strongly believed that it should be done by using parametric searching on the radius combined with the algorithm described in this paper, but we have not checked all details yet.

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