On the Number of Lines Tangent to Four Convex Polyhedra*

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June 28, 2002

Abstract

We prove that, under a certain general position assumption, the number of lines tangent to four bounded disjoint convex polyhedra in $\mathbb{R}^3$ with a total of n edges is $O(n^2)$. Under the same assumption, we show that a set of k bounded disjoint convex polyhedra has at most $O(n^2k^2)$ lines, possibly occluded, that are tangent to four of these polyhedra.

1 Introduction

The number of visibility events in a scene determines the complexity of many visibility-related problems arising in computer graphics, such as radiosity computation or hidden surface removal. Informaly, a visibility event corresponds to a combinatorial change in the view of a moving observer; such an event occurs when a viewing direction becomes tangent to some objects. Typically, a line in $\mathbb{R}^3$ can be tangent to up to four objects, unless they are in some kind of degenerate position. A key step in estimating the number of visibility events is to solve the following geometric problem:

> *G*iven k objects in $\mathbb{R}^3$, determine how many lines are tangent to 4 of them.

For k triangles, it is easy to see that the worst-case bound is $\Omega(k^4)$. In the case of 4 convex polygons with n vertices in total, the paper of Teller and Hohmeyer [4] proves, implicitly, that the number of tangents is $O(n^2)$.

This paper studies the case where the objects are convex, bounded, and disjoint polyhedra under some general position assumption. We first prove that 4 such objects, with a total of n edges, have $O(n^2)$ common tangent lines. We then extend this result, and show that k such polyhedra, with a total of n edges, have $O(k^2n^2)$ lines tangent to 4 of them, possibly intersecting the others.

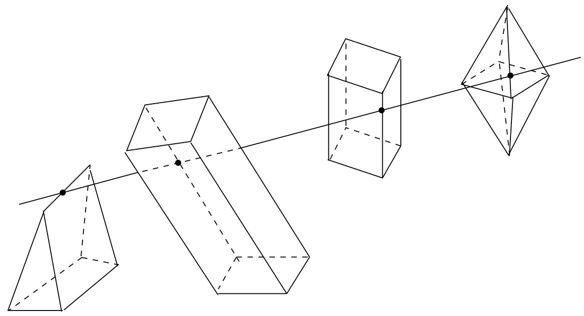

Figure 1: A line tangent to four convex polyhedra.

In order to define our notion of general position for polyhedra, we first define this notion for sets of lines. The following well-known fact is essential here and throughout the paper.

Fact 1 *Four lines in $\mathbb{R}^3$ admit either 0, 1, 2, or an infinite number of* transversals*, i.e., lines that intersect each of the four given lines (see e.g. [2, p. 164], [3]).*

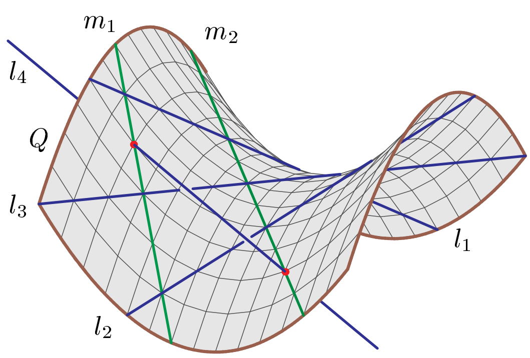

Figure 2: Two lines m_1 and m_2 meeting the four lines l_1, l_2, l_3, and l_4 (figure taken from [6]).

To give an idea why this is true, consider three pairwise skew lines l_1, l_2, and l_3 (see Figure 2). It is well

*This research was initiated at the McGill-INRIA Workshop on Computational Geometry in Computer Graphics, February 9-15, 2002, co-organized by H. Everett, S. Lazard, and S. Whitesides, and held at the Bellairs Research Institute of McGill University. This research was supported by INRIA, NSERC, FCAR and NSF.
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known [2, p. 15] that they always lie on a degree-two ruled surface $\mathcal{Q}$, namely a hyperbolic paraboloid or a hyperboloid of one sheet. In either case, there are two infinite families of pairwise skew lines that generate $\mathcal{Q}$. One family contains l_1, l_2, and l_3, and the other consists of all lines intersecting each of l_1, l_2, and l_3. In general, a fourth line l_4 meets $\mathcal{Q}$ in at most two points, and each of these points lies on exactly one line of each family. In particular they each lie on exactly one line of the second family (m_1 and m_2 in Figure 2). Thus there are, in general, at most two lines, m_1 and m_2, that meet each of l_1, l_2, l_3, and l_4. If l_4 lies on $\mathcal{Q}$, or if the lines l_1, l_2, and l_3 are not pairwise skew, there may be infinitely many lines meeting these four.

In the context of the study of common tangents to polyhedra, we wish to isolate as a degenerate case the situation of polyhedra having infinitely many common tangents. Thus we say that four lines are in *general position* if they admit at most 2 transversals, and that a set of convex polyhedra is in *general position* if every four edges, no two of which belong to the same polyhedron, are contained in four lines in general position.

To appreciate the impact of this general position assumption for convex polyhedra, we give a characterization of the degenerate sets of four lines. We omit here the proof, which is elementary. Four lines are not in general position if and only if (i) they belong to the same family of generators of a hyperbolic paraboloid or a hyperboloid of one sheet, (ii) two of the lines are coplanar, and the two other lines intersect in that plane, (iii) three of the lines are coplanar and the fourth line intersects that plane, or (iv) at least three of the four lines are concurrent. It follows from this characterization that sets of random polyhedra, defined as the convex hulls of random points, are in general position with probability one. On the other hand, boxes lying on the floor are not in general position.

Our main result is an upper bound on the number of lines tangent to four convex polyhedra in $\mathbb{R}^3$.

Theorem 2 *Four bounded, disjoint convex polyhedra in general position in $\mathbb{R}^3$, with n edges in total, have $O(n^2)$ common tangent lines. If one of the polyhedra has constant size, this bound improves to $O(n)$.*

When the number k of polyhedra is greater than 4, a direct application of the above theorem yields an upper bound of $O(n^2k^4)$, which can be improved as stated in the following theorem.

Theorem 3 *Given k bounded, disjoint convex polyhedra in general position in $\mathbb{R}^3$, with n edges in total, the number of lines tangent to four of the polyhedra is $O(n^2k^2)$. (The tangents may be occluded by some of the $k-4$ other polyhedra.)*

2 Proof of Theorem 2

We prove in this section the following proposition which directly yields Theorem 2.

Proposition 4 *Consider four bounded, disjoint convex polyhedra in general position in $\mathbb{R}^3$. Let P, Q, and R be three of these polyhedra, having p, q, and r edges, respectively, and let e be an edge of the fourth polyhedron. The number of lines intersecting e and tangent to P, Q, and R is $O(p+q+r)$.*

The proof proceeds as follows. We sweep the space with a plane rotating about the line l containing e. We define some critical events occurring during the sweep. We show that at and between two consecutive such events there are at most a constant number of tangent lines, and that the number of critical events is $O(p+q+r)$.

Let Π_t, $t \in [0, \pi]$, denote the parameterized sweep plane such that Π_t contains the line l for all t, and $\Pi_0 = \Pi_\pi$. Each plane Π_t intersects the three polyhedra P, Q, and R in three, possibly degenerate or empty, disjoint convex polygons, P_t, Q_t, and R_t (see Figure 3).

A line in plane Π_t that is tangent to P is either tangent to P_t or properly intersects P_t, in which case P_t must be a face or an edge of P lying in Π_t, implying Π_t is tangent to P. We thus have the following simple lemma.

Lemma 5 *Any line tangent to P, Q, and R that intersects e is contained in some plane Π_t such that either Π_t is tangent to P, Q, or R, or the line is tangent to all three polygons P_t, Q_t, and R_t in that plane Π_t.*

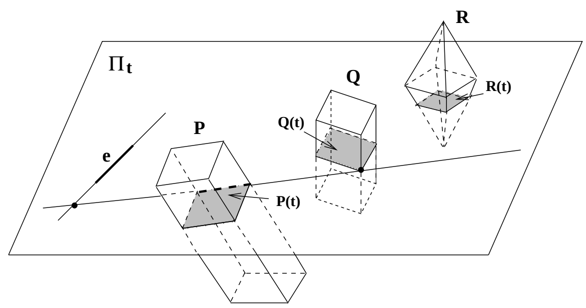

Figure 3: Plane Π_t contains edge e and intersects polyhedra P, Q, and R in polygons P_t, Q_t, and R_t, respectively. In this figure Π_t is in critical F-position.

The sweep planes Π_t that are tangent to P, Q or R are called *tangent planes.*

Lemma 6 *There are at most 24 lines tangent to P, Q, and R and intersecting e that are distinct from l and lie in the tangent planes.*

Proof: Let $\Pi^\star$ denote a tangent plane. It intersects P, Q, and R in three possibly degenerate convex polygons, $P^\star$, $Q^\star$, and $R^\star$. At most one of these, say $P^\star$, is precisely an edge or a face of one of the polyhedra; otherwise, one could choose three coplanar edges from distinct polyhedra (edge e and two others), leading to a contradiction of the general position assumption. Thus a line in $\Pi^\star$ that is tangent to P, Q, and R has to be tangent to at least two polygons, $Q^\star$ and $R^\star$. Thus $\Pi^\star$ contains at most four lines tangent to P, Q and R.

We now count the number of relevant tangent planes. If the intersection of $\Pi^\star$ with P is an edge or a vertex of P lying on l, then a line in $\Pi^\star$ that is tangent to P and intersects e is necessarily l, since $e \cap P = \emptyset$. Thus we need only to consider tangent planes that do not intersect P along l. At most two such sweep planes are tangent to P. A similar argument applies to Q and R. Thus there are at most six such tangent planes in total, which implies the result. □

We now count the number of lines tangent to all three polygons P_t, Q_t, and R_t, for t ranging over $[0, \pi]$. The number of such tangents is equal to the number of times a bitangent to the polygons P_t and Q_t coincides with a bitangent to the polygons Q_t and R_t, for $t \in [0, \pi]$. (A bitangent is a line tangent to two polygons; its contact points with polygon vertices are called support vertices.)

Polygons P_t, Q_t, and R_t have vertices determined by the intersections of Π_t with the edges of the polyhedra. As the sweep plane rotates, these polygons deform. The edges of the polyhedra P, Q, and R intersected by Π_t do not change except when Π_t passes through a vertex of a polyhedron. However the set of polyhedron edges containing the support vertices of a bitangent to two polygons in Π_t may change even though Π_t does not pass through a vertex of a polyhedron (see Figures 3 and 4).

We define a set of *critical positions* of the plane Π_t such that in the open interval between consecutive critical positions, the sets of polyhedron edges containing the support vertices of the bitangents to polygons P_t and Q_t, and the support vertices of the bitangents to Q_t and R_t, do not change. We define two types of critical positions: *critical V-positions* and *critical F-positions.*

Plane Π_t is at a *critical V-position* if it goes through a vertex of P, Q, or R. As the plane rotates between two consecutive critical V-positions, the vertices of the polygons P_t, Q_t, and R_t stay on the same polyhedron edges.

Plane Π_t is at a *critical F-position* relative to polyhedron P if it contains a line distinct from l that

(i) goes through some face f of P,

(ii) is tangent to one of the polygons $Q \cap \Psi$ or $R \cap \Psi$, where Ψ is the plane determined by f, and

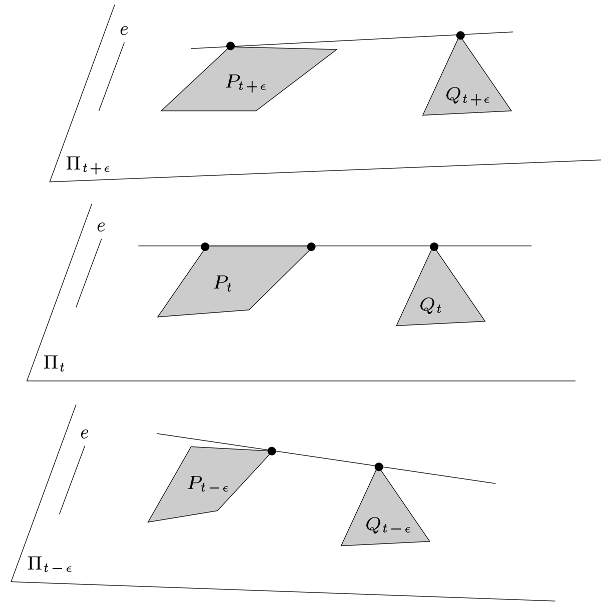

Figure 4: The plane Π_t is at a critical F-position. The planes $\Pi_{t-\epsilon}$ and $\Pi_{t+\epsilon}$ show the situation shortly before and after the critical F-position. The set of polyhedron edges containing the support vertices of the bitangent changes at Π_t.

(iii) goes through $\Psi \cap l$ if this intersection is a point, and otherwise is parallel to l.

Critical F-positions relative to Q and R are defined similarly. A plane Π_t is at a *critical F-position* if it is at a critical F-position relative to polyhedron P, Q or R. A *critical position* is a critical V-position or a critical F-position. We now prove in the following lemma that the critical positions have the desired property.

Lemma 7 *In the open interval between consecutive critical positions, the sets of polyhedron edges containing the support vertices of the bitangents to any two of the polygons P_t, Q_t, and R_t do not change.*

Proof: Consider a plane Π_{t^*} such that the set of polyhedron edges containing the support vertices of a bitangent m to two polygons, say without loss of generality P_{t^*} and Q_{t^*}, changes for t in any neighborhood of t^*. We prove that Π_{t^*} is necessarily at a critical position.

We consider first the case $m \neq l$. If m goes through a vertex of P or Q, then Π_{t^*} is at a critical V-position. Suppose now that m does not go through a vertex of P or Q. Then the bitangent m

(a) contains an edge of P_{t^*} or Q_{t^*}, say P_{t^*},

(b) is tangent to Q_{t^*}, and

(c) intersects (or is parallel to) the line l, since l also lies in Π_{t^*}.

It follows that m

(a′) goes through a face of P,

(b$'$) is tangent to Q, and

(c$'$) intersects (or is parallel to) the line l.

Let Ψ denote the plane containing the face of P in question. Condition (b$'$) implies that m is either tangent to the polygon $Q \cap \Psi$ or properly intersects it, in which case $Q \cap \Psi$ is a face or an edge of Q. If m is tangent to the polygon $Q \cap \Psi$, then Π_{t^*} is at critical F-position by definition. Suppose now that m properly intersects $Q \cap \Psi$. In Ψ, there are infinitely many lines χ satisfying conditions (a$'$), (b$'$) and (c$'$) and intersecting $Q \cap \Psi$. Each of the lines χ lies in a sweep plane Π_t and thus has the form $\chi_t = \Psi \cap \Pi_t$. Note that $m = \chi_{t^*}$.

Line l does not lie in Ψ because otherwise e, an edge of P, and an edge of Q would be coplanar, leading to a contradiction of the general position assumption. Thus l intersects Ψ in at most one point and, by (c$'$), all the lines χ_t pass through this point (or are parallel to l). Thus the lines χ_t form in Ψ a double-wedge or a strip bounded by two extremal lines. It follows that the lines χ_t are defined for t in an interval, say $[t_0, t_1]$. Each of the two extremal lines, χ_{t_0} and χ_{t_1}, goes through a vertex of $\Psi \cap P$ or $\Psi \cap Q$ that is also a vertex of P or Q. Thus the planes Π_{t_0} and Π_{t_1} are at critical V-positions.

Note that $t^* \in [t_0, t_1]$ since m is one of the lines χ_t. If $t^* = t_0$ or t_1, then Π_{t^*} is at a critical position. Otherwise $t^* \in (t_0, t_1)$, and as t ranges over any small enough neighborhood of t^*, the set of polyhedron edges containing the support vertices of the bitangent χ_t does not change, although χ_t has more than two support vertices. This contradicts the definition of Π_{t^*}.

Finally, we consider the case where $m = l$. Then l is a bitangent to P_t and Q_t in all planes Π_t and its support vertices always stay on the same polyhedron edges, for all Π_t. Thus the set of polyhedron edges containing the support vertices of the bitangent $m = l$ does not change for any t. This again contradicts the definition of Π_{t^*}. □

Lemma 8 *There are at most 16 lines tangent to all three polygons P_t, Q_t, and R_t in planes Π_t ranging over the open interval between two consecutive critical positions.*

Proof: In any plane Π_t, for each of the at most four bitangents between P_t and Q_t, at most two bitangents between Q_t and R_t can share the same support vertices on Q_t. Each of those at most eight pairs of bitangents corresponds to a quadruple of polyhedron edges which admits at most two transversals, by Fact 1 and the general position assumption. This concludes the proof since by Lemma 7, in the open interval between consecutive critical positions, the support vertices of the bitangents to polygons P_t and Q_t, and of the bitangents to Q_t and R_t, stay on the same polyhedron edges. □

Lemma 9 *In any plane Π_t, there are at most 4 lines tangent to all three polygons P_t, Q_t, and R_t.*

Proof: The lemma is obvious since any tangent to all three polygons is also a tangent to two of them. □

Lemma 10 *There are at most $5\,(p+q+r)$ critical positions.*

Proof: The number of critical V-positions is at most the total number of vertices of P, Q, and R, and hence is less than $p+q+r$, the total number of edges of P, Q, and R.

We now count the number of critical F-positions. A plane Ψ supporting a face of P contains at most four lines that go through point $\Psi \cap l$, or are parallel to l, and that are tangent to $Q \cap \Psi$ or $R \cap \Psi$. Thus a face of P generates at most four critical F-positions. A similar argument holds for the faces of Q and R.

Thus the total number of critical F-positions for a given edge e is at most four times the number of faces of P, Q, and R, and thus is at most $4\,(p+q+r)$, since the number of faces of a polyhedron is at most the number of its edges. □

We can now conclude the proof of Proposition 4. By Lemma 10, there are at most $5\,(p+q+r)$ critical positions. Thus, by Lemmas 8 and 9, there are at most $(16+4)\,5\,(p+q+r)$ lines tangent to all three polygons P_t, Q_t, and R_t, as t ranges over $[0, \pi]$. On the other hand, by Lemma 6, there are at most 24 lines tangent to P, Q, and R that lie in the tangent planes. Thus, by Lemma 5, there are at most $100\,(p+q+r) + 24$ lines intersecting e and tangent to P, Q, and R.

3 Proof of Theorem 3

Consider k polyhedra $P_1, \ldots, P_k$, and let n_i denote the number of edges of P_i. Choose an edge e, and let P_j, P_l, and P_m be distinct polyhedra not containing edge e. From Proposition 4, we know that the number of tangents to P_j, P_l, and P_m intersecting e is no more than $C(n_j + n_l + n_m)$, where C is some constant. We sum, over all edges e, the number of tangents intersecting e. There are n edges in the scene, so the number T of tangents to four polyhedra satisfies

$$T \leqslant n \sum_{j<l<m} C\,(n_j + n_l + n_m).$$

Since each n_i, $1 \leqslant i \leqslant k$, appears $\binom{k-1}{2}$ times in the sum, it follows that

$$T \leqslant C\,n \sum_{1 \leqslant i \leqslant k} n_i \binom{k-1}{2} = C\,n^2 \binom{k-1}{2}$$

so T is $O(n^2 k^2)$ as claimed.

4 Discussion and open problems

We have presented bounds on the number of lines tangent to four polyhedra. The proofs are inspired by a method which was, to our knowledge, first used in [1] (see also Schiffenbauer's survey [5]).

We believe our results generalize to polyhedra that are not pairwise disjoint, and that our proofs can easily be transformed into an $O(n^2 \log n)$ time algorithm for computing the lines tangent to four polyhedra. Furthermore, we have constructions that give matching lower bounds for our upper bounds. The final version of this paper should contain these extensions.

We conclude with the following open problem. The bound of Theorem 3 is only for lines tangent to four polyhedra among k. Ideally, we would like to bound the number of tangent lines such that the shortest line segment spanning all four points of contact does not intersect any other polyhedron. Is it possible to get a better bound for this kind of non-occluded visibility event?

Acknowledgements. We thank the other workshop participants for stimulating conversations and for creating a productive atmosphere.

References

[1] M. de Berg, D. Halperin, M. Overmars, and M. van Kreveld, "Sparse arrangements and the number of views of polyhedral scenes," in *Internat. J. Comput. Geom. Appl.*, vol. 7, no. 3, 1997, pp. 175-195.

[2] D. Hilbert and S. Cohn-Vossen. Geometry and the Imagination. Chelsea Publishing Company, New York, USA, 1952.

[3] M. Hohmeyer and S. Teller, "Determining the lines through four lines," *Journal of Graphics Tools*, vol. 4, no. 3, pp. 11-12, 1999.

[4] M. Hohmeyer and S. Teller, "Stabbing Oriented Convex Polygons in Randomized $O(n^2)$ Time," *CONM*, vol. 178, Proc. Jerusalem Combinatorics '93, American Mathematical Society, Providence RI, USA, pp. 311–318.

[5] R. Schiffenbauer, "A Survey of Aspect Graphs", Technical Report TR-CIS-2001-01, Dept. of Computer and Information Science, Polytechnic University, 2001.

[6] F. Sottile. `http://www.maths.univ-rennes1.fr/~raag01/surveys/ERAG/S4/1.html`.