Constructing Differentiable Homeomorphisms between Isomorphic Triangulations

Frederick Crimins* Diane Souvaine*

September 15, 2002

1 Introduction

Let $P = \{p_1, p_2, \ldots, p_n\}$ and $Q = \{q_1, q_2, \ldots, p_n\}$ be two sets of triangulated points in the plane such that every triangle $p_i p_j p_k$ corresponds to a triangle $q_i q_j q_k$, *i.e.* the triangulations are isomorphic. We aim to construct a differentiable homeomorphism between the triangulated areas defined by P and Q.

A *homeomorphism* is a one-to-one, continuous mapping of R^d onto R^d with a continuous inverse. These continuous mappings are used in such fields as computer graphics for mapping a rectangular image onto a smooth or triangulated object. For this purpose, piecewise linear homeomorphisms are generally used [2, 3, 9]. However, since they are not differentiable, they induce irregularities along the edges of the triangles to which they are applied; *i.e.* a piecewise linear homeomorphism has order-0 (C^0) continuity on the boundaries between the linearly transformed pieces. Therefore, a homeomorphism with higher order continuity, a differentiable homeomorphism, is preferable for such applications. A *differentiable homeomorphism* is a one-to-one, differentiable mapping of R^d onto R^d with a differentiable inverse.

We attempt to generate such a mapping between two isomorphically triangulated polygons by defining a C^1 grid on each triangle that maintains its C^1 continuity along a shared edge between two triangles. Thus, all grid lines pass with C^1 continuity over all shared edges (Figure 1). More specifically, these grid lines perpendicularly intersect the edges of the triangles. We use this grid to define a new coordinate system, the triangular coordinate system.

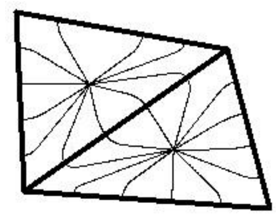

Figure 1: C^1 grid lines maintain their differentiability when passing over edges shared by adjacent triangles.

Note, however, that no grid line through a vertex of a triangle can pass over both of the intersecting triangle edges with C^1 continuity. To do so, the grid line would have to be perpendicular simultaneously to both edges incident to the vertex, which is impossible. As a result of the C^0 continuity of the grid lines at triangle vertices, the homeomorphism suggested in this paper will not be differentiable at these points. Similar finite subsets of points with reduced continuity, such as the vertices in a Delauney triangulation, arise in surface interpolation [1, 4, 6, 8, 10]. Given two planar point sets, two simple polygons, or two polygons with holes, each defined on m points, however, producing isomorphic triangulations of the input sets may require the addition of as many as $\Theta(n^2)$ vertices [2, 3, 9].

Even when we reconcile ourselves to reduced continuity at triangle vertices, three drawbacks to our approach remain. First, the centroid of the triangle is used as the source point for the grid lines, so

*EECS Department, Tufts University, Medford, MA 02155, ¡fcrimins,dls¿@eecs.tufts.edu. Research supported in part by NSF grant EIA-99-96237

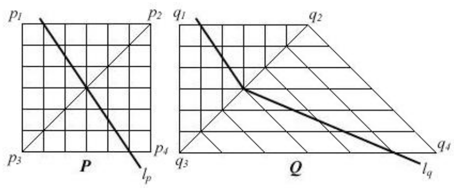

Figure 2: The effect of a piecewise linear homeomorphism mapping P onto Q. The line l_p gets mapped to l_q. Triangle $p_1p_2p_3$ is isomorphic to $q_1q_2q_3$, and triangle $p_2p_3p_4$ is isomorphic to $q_2q_3q_4$.

for some obtuse triangles this construction fails. Second, the computation of the triangular coordinates of a point involve solving a cubic polynomial, which is subject to robustness issues; *i.e.* calculation errors are frequent due to imprecise floating point arithmetic. Finally, there is not yet any proof of the differentiability of this construction one the region bounded by the convex hull of the point set and excluding triangle vertices.

In the following section, we present the homeomorphism which we conjecture to be differentiable. We explain the reasoning that spawned the construction and the method of generating the actual homeomorphism. In Section 3 we highlight a subset of obtuse triangles on which the homeomorphism is undefined, while noting that obtuse triangles can cause trouble in other domains as well (*e.g.* mesh generation [5]). The remaining sections describe our current implementation of the homeomorphism and directions for future work.

2 A New Coordinate System

When a piecewise linear homeomorphism is applied to a triangulated polygon, each triangle undergoes a unique linear transformation in which the transformed image of a line that crosses over an edge shared by two triangles will remain continuous but not necessarily differentiable. For example, in Figure 2, a piecewise linear homeomorphism mapping P onto Q will map the line l_p to the line l_q. Clearly, l_q is not differentiable at the point where it crosses over the edge shared by Q's two triangles.

It is easier to see the full effect of the homeomorphism by examining the grid lines in P as they also lose differentiability when mapped onto Q. A coordinate system that does not maintain smooth grid lines clearly cannot support a globally differentiable mapping. Resolving this shortcoming is the basis of our new coordinate system, in which we will maintain smooth grid lines between adjacent triangles. More precisely, we will require that every grid line be perpendicular to the edge of a triangle at its point of intersection with that edge. However, this is not a simple construction, mainly because it is unclear how to make all of the grid lines perpendicular to triangle edges even as they approach a vertex of a triangle.

Our solution to this problem is based on the following construction (see Figure 3). First find the centroid of the triangle. This point is the intersection of the three segments between each vertex and the midpoint of the opposite edge. From the centroid, construct line segments, l_A, l_B, and l_C, perpendicular to each of the three edges of the triangle. We explain later why these segments, called *center segments*, are actually grid lines.

Notice that it is possible to construct a triangle for which one of these segments extends outside the triangle before it intersects the line containing its respective edge. For now, we will assume that this is not the case. This limitation is discussed further in Section 3.

Next, construct line segments, which we will call *wing segments*, from each triangle vertex to the midpoint of both visible center segments that lie on either side of that vertex (s_{A1}, s_{A2}, s_{B1}, s_{B2}, s_{C1}, and s_{C2}). These nine segments, three center segments and six wing segments, will be known as *contour guides*. We use these contour guides to construct the grid lines of our coordinate system.

Each grid line or *contour* consists of a straight line segment c and a circular arc a. The segment c emanates from the centroid of the triangle as if it were a polar grid line until it intersects a wing segment (*e.g.* s_{A2}). The arc a is then constructed tangent to the line containing c at its point of intersection with the wing segment; and from there, a curves toward the edge of the triangle so that it is perpendicular to that edge. Examples of these grid lines can be seen in Figure 1. The three center segments are degenerate

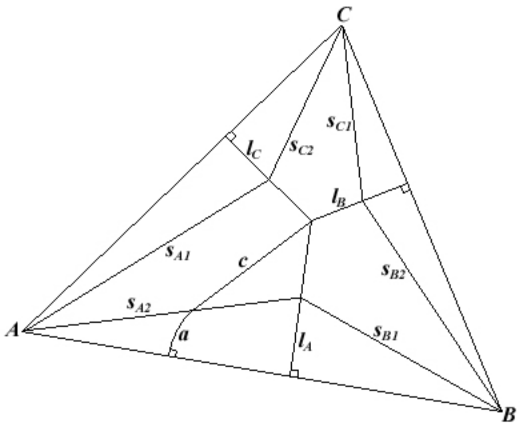

Figure 3: The contour guides aid the construction of a contour.

cases of grid lines because the arcs in these contours are straight line segments. Notice that as these composite grid lines approach a vertex of the triangle, the radius of the resulting arc approaches zero.

All but three of the contours remain perpendicular as they intersect the edges of the triangle. The three degenerate contours that do not maintain this property are those that go through the vertices of the triangle. We conjecture the differentiability of our homeomorphism on the C^1 continuity of these grid lines. Since the continuity of the grid lines at the vertices of the triangles is merely C^0 we note that differentiability at these points is not maintained.

We use these contours to specify the coordinates (e, p, c) of a point X within a triangle. We call this coordinate system the *triangular coordinate system.* In a set of triangular coordinates, the e-coordinate identifies the edge of the triangle intersected by the contour through X. The set of edge labels, 1, 2, and 3, is therefore a sufficient domain for e. The p-coordinate may assume any value between 0 and 1 and it indicates the position at which X's contour intersects edge e. This value is specified as a proportion of the total length of e, *e.g.*, if $p = 0.5$, then X's contour will intersect e at the midpoint of e. The c-coordinate also assumes a value on the $[0, 1]$ interval. This coordinate identifies X's distance from the centroid as measured along the contour of X and as a proportion of the length of this contour. For example, if X lies on an edge of the triangle then $c = 1$ and if X is the centroid of the triangle then $c = 0$.

By use of triangular coordinates rather than Cartesian coordinates, we hypothesize that a differentiable homeomorphism can be defined between two isomorphically triangulated polygons P and Q. For a point p in P, first find the bounding triangle of p in P's triangulation. Then, with respect to this triangle, determine the triangular coordinates of p. Finally, map the image of p in the corresponding isomorphic triangle in Q's triangulation using the same triangular coordinates as p. Clearly this process also requires that the edge labellings in isomorphic triangles are isomorphic. Furthermore, as noted above, this differentiability breaks down at the vertices of the triangles where the C^1 continuity of the grid lines is not maintained.

Although this may seem like a simple process, the complexity of one step has been severely overlooked: determining the triangular coordinates of a point X is not a trivial task even with an acute triangle as in Figure 3. If X lies within the hexagon formed by the six wing segments of the triangle, the calculation is simple because two points on the line segment portion of the contour are known, namely X and the centroid. However, if X lies to the outside of this hexagon, then this computation is much more difficult, mainly because none of the properties of the contour are readily obvious. Finding the contour in this case involves solving a cubic polynomial. Due to space constraints, a full explanation of this complex calculation has been excluded from this paper.

3 Degenerate Cases

As mentioned in the last section, this construction cannot maintain differentiability at the vertices of the triangles. However, there exists a more substantial limitation: in certain obtuse triangles this construction cannot be applied. It is possible to construct a triangle in which one of the edge-perpendiculars drawn from the centroid extends outside the triangle before it intersects the line containing its respective edge. For example, see Figure 4.

This flaw prompted us to try choosing a number of

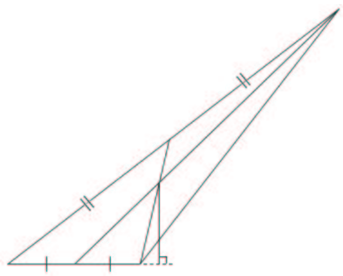

Figure 4: The perpendicular constructed from the centroid to the lower edge of this triangle extends outside of the triangle before intersecting the line containing that edge. Therefore, triangular coordinates cannot be applied to this triangle.

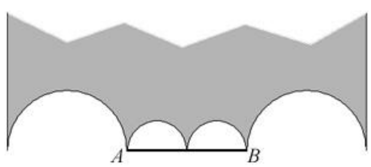

Figure 5: Given the fixed points A and B, our differentiable homeomorphism can be applied to $\triangle ABC$ if and only if C lies in the shaded region, which extends upwards.

different center points, such as the incenter, for which the edge-perpendicular segments are completely contained in the triangle. Each of the homeomorphisms produced by alternate center points, however, were non-differentiable. We conjecture that no similarly defined homeomorphism using a center point other than the centroid will be differentiable.

In general, there are two types of triangles in which the contour guides cannot properly be constructed. Triangles containing grossly obtuse angles often cause this dysfunctionality; and in addition, triangles that contain an obtuse angle between a very small and a very large side are also susceptible. However, it is easy to show that this construction can be successfully applied to all acute triangles and all isosceles triangles. Figure 5 displays the set of valid triangles under this construction.

We have also shown that it is not possible to deconstruct a discordant triangle into usable triangles with the help of additional interior points or Steiner points. The complexity of this process is compounded by the requirement that additional points would be inserted into a discordant triangle's isomorphic triangle as well. Isomorphic triangulations that avoid triangles of this type are of great interest.

Achieving an optimal triangulation by minimizing the maximum angle in a triangulated mesh has been the source of extensive research. Lawson [7] has shown that the Delauney triangulation achieves this property in two dimensions. In addition, Bern and Eppstein [5] discuss optimal triangulations while examining various formulations of optimality.

4 Implementation

We have implemented this homeomorphism in C++ using the Library of Efficient Data Types and Algorithms (LEDA) version 4.1, maintained by Algorithmic Solutions. Our implementation can be found at http://www.eecs.tufts.edu/r/geometry/dhm. Our program generated the images shown in Figure 6 by applying the mapping described above between the two isomorphically triangulated polygons. The program first calculated the triangular coordinates from the Cartesian coordinates of the points along a series of vertical parallel lines in the first polygon. It then used those same triangular coordinates to compute the Cartesian coordinates of the points in the second polygon. As you can see, the image of the parallel lines is smooth as it crosses triangle boundaries.

A drawback of this implementation is that rounding and calculation errors frequently occur when converting points from Cartesian to triangular coordinates.

5 Further Research

This paper leaves one major question unanswered: how might one prove that homeomorphism presented in this paper is differentiable? The conversion of a triangular coordinate system to a Cartesian coordinate system is rather complex; it is far more complex than the conversion from polar to Cartesian. As a result, the formulation of a function obtained using the triangular coordinate system in terms of Cartesian or polar coordinates is also unclear. In addition,

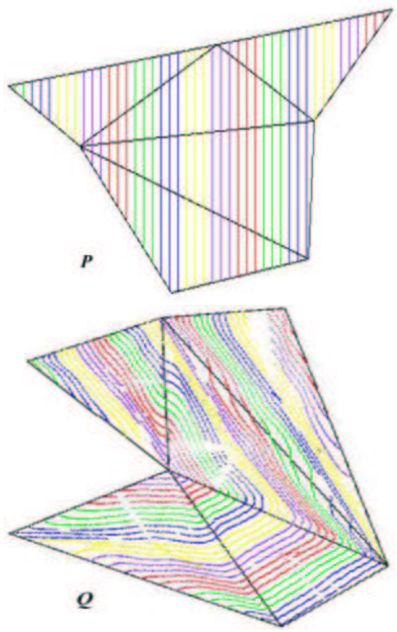

Figure 6: The effect of a homeomorphism mapping P onto Q using triangular coordinates.

the apparent requirement that the centroid be used as the center point for the coordinate systems is not fully understood. These issues contribute substantially to the difficulty of proving the differentiability over triangular boundaries of this homeomorphism. Methods similar to those used for analyzing primal-dual relationships have been suggested, however further insights are needed.

A second question is more interesting. Given two planar point sets, two simple polygons, or two polygons with holes, each defined on m points, isomorphic triangulations of the input sets can be produced, adding at most $\Theta(n^2)$ vertices [2, 3, 9]. Can non-obtuse isomorphic triangulations be produced? At a cost of how many additional vertices?

6 Acknowledgements

The current version of the paper benefits from the careful reading, insights, and observations of the anonymous referees as well the comments of Marcia Lazo. We thank them.

References

[1] N. Amenta, M. Bern, and M. Kamvysselis. A new Voronoi-based surface reconstruction algorithm. In *Proceedings of the 25th Annual Conference on Computer Graphics*, pages 415–421, 1998.

[2] B. Aronov, R. Seidel, and D. Souvaine. On compatible triangulations of simple polygons. *Computational Geometry: Theory and Applications*, pages 27–35, 1993.

[3] M. Babikov, D. Souvaine, and R. Wenger. Constructing piecewise linear homeomorphisms of polygons with holes. *Proceedings of the 9th Canadian Conference on Computational Geometry*, pages 6–10, 1997.

[4] C.L. Bajaj and I. Ihm. Smoothing polyhedra using implicit algebraic splines. In *Proceedings of the 19th Annual Conference on Computer Graphics*, pages 79–88, 1992.

[5] M. Bern and D. Eppstein. Mesh generation and optimal triangulation. In D. Zhu and F. Hwang, editors, *Computing in Euclidean Geometry, World Scientific, Lecture Notes Series on Computing*, volume 1. 1992.

[6] H. Hiyoshi and K. Sugihara. Voronoi-based interpolation with higher continuity. In *Proceedings of the 16th Annual Symposium on Computational Geometry*, pages 242–250, 2000.

[7] C.L. Lawson. Software for C^1 surface interpolation. In J. Rice, editor, *Mathematical Software III*, pages 161–194. Academic Press, 1977.

[8] R. Sibson. A brief description of natural neighbour interpolation. *Interpreting Multivariate Data*, pages 21–36, 1981.

[9] D. Souvaine and R. Wenger. Constructing piecewise linear homeomorphisms. *DIMACS Technical Report*, 94-52, 1994.

[10] A.H. Thiessen. Precipitation averages for large areas. *Monthly Weather Review*, pages 1082–1084, 1911.