MATH 1410
Solutions for Homework 1
Submitted Friday, January 18

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(1) Solve the following systems of equations:

(a) $\begin{cases} 5x + y = 59 \\ x + 5y = 31. \end{cases}$

Solution:

There are *many* ways to solve this. Here are *four:*

Method 1: Substitution (solve for x in the second equation):

We use subtraction to isolate the x in the second equation:

$$x+5y=31 \implies x+5y-5y=31-5y \implies x=31-5y.$$

Next, we substitute this formula into the x in the first given equation and isolate y:

$$5x+y=59 \implies 5(31-5y)+y=59 \implies 155-25y+y=59 \implies 155-24y=59$$

$$\implies -24y=59-155 \implies -24y=-96 \implies \frac{-24y}{-24}=\frac{-96}{-24} \implies y=4.$$

Lastly, we substitute this value into the y in our formula for x:

$$x = 31-5y = 31-5(4) = 31-20 = 11.$$

So, the only solution for the given system is $\boxed{(11, 4).}$

(continued)

(continued) **<u>Method 2</u>: Substitution** (solve for y in the first equation):

We use subtraction to isolate the y in the first equation:

$$5x+y=59 \implies 5x+y-5x=59-5x \implies y=59-5x.$$

Then, we substitute this formula into the y in the second given equation and isolate x:

$$x+5y=31 \implies x+5(59-5x)=31 \implies x+295-25x=31 \implies 295-24x=31$$

$$\implies -24x=31-295 \implies -24x=-264 \implies \frac{-24x}{-24}=\frac{-264}{-24} \implies x=11.$$

Finally, we substitute this value into the x in our formula for y:

$$y = 59-5x = 59-5(11) = 59-55 = 4.$$

Thus, the only solution for the given system is $\boxed{(11, 4).}$

<u>Method 3</u>: Elimination (eliminate the x):

We will create new equations from the given ones. To avoid confusion, let us number them, starting with the originals:

$$\begin{array}{llcrcl} ① & 5x & + & y & = & 59 \\ ② & x & + & 5y & = & 31 \end{array}$$

To begin, we multiply both sides of ② by 5:

$$5(x+5y)=5(31) \implies ③\ 5x+25y=155.$$

Next, we subtract ③ from ①.†

$$①-③:\ (5x+y)-(5x+25y)=59-155 \implies 5x+y-5x-25y=-96$$

$$\implies -24y=-96 \implies y=\frac{-96}{-24}=4.$$

We now plug this y-value into ① or ② (say, ①):

$$5x+y=59 \implies 5x+4=59 \implies 5x=55 \implies x=11.$$

Ergo, the only solution for the given system is $\boxed{(11, 4).}$ (continued)

†Just to be clear, we're not really *subtracting equations*. For example, if $A=B$ and $C=D$, then $(A=B)-(C=D)$ doesn't make sense! On the other hand, if A and B are equal, then $A-C=B-C$, and if C and D are equal, then, $B-C=B-D$. Combining these results, if $A=B$ and $C=D$, then it follows that $A-C=B-D$; "subtracting the second equation from the first" is just the act of writing this third equation. Similarly, "adding the two equations" is just the act of writing $A+C=B+D$.

(continued) **Method 4: Elimination** (eliminate the y)

We will create new equations from the given ones. To avoid confusion, let us number them, starting with the originals:

$$\begin{array}{rrcrcl} ① & 5x & + & y & = & 59 \\ ② & x & + & 5y & = & 31 \end{array}$$

To begin, we multiply both sides of ① by 5:

$$5(5x+y) = 5(59) \implies ③\ 25x+5y = 295.$$

Next, we subtract ② from ③.

$$③ - ②: \ (25x+5y)-(x+5y) = 295-31 \implies 25x+5y-x-5y = 264$$

$$\implies 24x = 264 \implies x = \frac{264}{24} = 11.$$

We now plug this x-value into ① or ② (say, ①):

$$5x+y = 59 \implies 5(11)+y = 59 \implies 55+y = 59 \implies y = 4.$$

Hence, the only solution for the given system is $\boxed{(11, 4).}$

(b) $\begin{cases} 2x + 3y = 1 \\ x - y = 10. \end{cases}$

Solution:

Let's use substitution to solve this; specifically, let us isolate x in the second equation:

$$x-y = 10 \implies x = 10+y.$$

Then, we substitute this formula into the x in the first equation and solve for y:

$$2x+3y = 1 \implies 2(10+y)+3y = 1 \implies 20+2y+3y = 1 \implies 5y = -19y = -\frac{19}{5}.$$

Lastly, we plug this y-value into our formula for x:

$$x = 10+y = 10+\left(-\frac{19}{5}\right) = \frac{10}{1}-\frac{19}{5} = \frac{50}{5}-\frac{19}{5} = \frac{50-19}{5} = \frac{31}{5}.$$

Therefore, the solution of the given system is $\boxed{\left(\frac{31}{5}, -\frac{19}{5}\right)} = (6.2, -3.8).$

(c) $\begin{cases} a + b + c = 1 \\ a + 2c = 5 \\ 2a + 2b + c = 0. \end{cases}$

Solution:

To do substitution with *three* equations in *three* unknowns, we isolate one of the variables in one of the equations, then substitute its formula into *both* of the other equations; this will produce two equations in two unknowns, which we know how to solve. The final step will be to calculate the value of the variable that we isolated at the beginning.

Let us isolate a. Of the three equations, the second one will yield the simplest formula for it:

$$a+2c=5 \implies a=5-2c.$$

We now substitute this into the first equation:

$$a+b+c=1 \implies (5-2c)+b+c=1 \implies ① \; b-c=-4.$$

Next, we substitute it into the third equation:

$$2a+2b+c=0 \implies 2(5-2c)+2b+c=0 \implies 10-4c+2b+c=0 \implies ② \; 2b-3c=-10.$$

① and ② form a *new* system that we can solve by, say, isolating b in ①:

$$b-c=-4 \implies b=c-4.$$

We then substitute this into ② and solve for c:

$$2b-3c=-10 \implies 2(c-4)-3c=-10 \implies 2c-8-3c=-10 \implies -c=-2 \implies c=2.$$

Now we can compute the values of the other variables:

$$b = c-4 = 2-4 = -2, \quad \text{and} \quad a = 5-2c = 5-2(2) = 5-4 = 1.$$

Consequently, the only solution of the given system is $\boxed{(1, -2, 2).}$

(2) Find three triples (x, y, z) so that $\begin{cases} x + y + z = 0 \\ 2x + 3y + 4z = 1. \end{cases}$

Solution:

Let us ignore the fact that this system has fewer equations than unknowns and try to solve it using substitution. To begin, $x+y+z=0 \implies x=-y-z$.

Then, $2x+3y+4z=1 \implies 2(-y-z)+3y+4z=1 \implies -2y-2z+3y+4z=1$

$\implies y+2z=1 \implies y=1-2z.$

To get a fixed value for z, we would need a third equation. Because there are *no* equations left, z can take *any* value! In other words, we can make z equal to whatever we want!

Let t represent the value that we pick for z. Then, $y=1-2z \implies y=1-2t$.

Furthermore, $x = -y-z = -(1-2t)-t = -1+2t-t = -1+t = t-1$.

So, the solution that we would get is $(t-1, 1-2t, t)$. If we want *three* triples, then we should pick *three* values for t, like 0, 1, and 2:

$$\boxed{(-1, 1, 0), \ (0, -1, 1), \ \text{and} \ (1, -3, 2).}$$

Of course, these needn't be our only choices. Here are three more triples that satisfy the given equations:

$$\boxed{(2, -5, 3), \ (3, -7, 4), \ \text{and} \ (4, -9, 5).}$$

And here are three more:

$$\boxed{(-2, 3, -1), \ (-0.5, 0, 0.5), \ \text{and} \ \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right).}$$

(3) John, Jeff, and Jessica are going on a road trip. John spends three times as much as Jeff and Jessica together. Jeff brought 100 dollars more than Jessica, but had the same amount money [sic] left over at the end. Jessica, on the other hand, spent only half as much money as Jeff. How much money did each of them spend on this trip?

Solution:

Let a, b, and c (respectively) be the number of dollars that John, Jeff, and Jessica (respectively) spent on the trip. Since John spent three times as much as Jeff and Jessica together,

$$a = 3(b+c) = 3b+3c.$$

Because Jessica spent half as much as Jeff,

$$c = \frac{b}{2}, \quad \text{or} \quad b = 2c.$$

Now for the third sentence: "Jeff brought 100 dollars more than Jessica, but had the same amount [of] money left over at the end." On first reading, one might think that we need two more variables: one for the amount of money that Jeff *brought* with him on the trip, and one for the amount that Jessica brought. After some consideration, though, we realize something: the only way that Jeff and Jessica can have the same amount left over at the end is if Jeff *spent* 100 dollars more than Jessica! In other words, $b = 100+c$. Substituting this into the last equation above, we get

$$b = 2c \implies 100+c = 2c \implies -c = -100 \implies c = 100.$$

Then, $b = 2c = 2(100) = 200$, and $a = 3(200)+3(100) = 600+300 = 900$.

Accordingly, John spent \$900, Jeff spent \$200, and Jessica spent \$100.

(4) Find the roots of the following polynomials.

(a) $x^2+2x-3=0$

Solution:

A *root* is a number that when substituted into the variable in a polynomial makes that polynomial equal to zero. There is a LOT to do before we can identify the roots of the given polynomial.

First, we observe that for any number k,

$$(x+k)^2 = (x+k)(x+k) = x(x+k)+k(x+k) = x^2+xk+kx+k^2 = x^2+2kx+k^2.$$

This is an example of a *perfect square.*

(continued)

(continued) We now notice that the first two terms in the given polynomial, x^2+2x, are two of the three terms in a perfect square. Which one? Well, we need $2k=2$, so $k=1$. Consequently, the missing term is $k^2=1^2=1$, and together, these three terms are the square of $x+k=x+1$; that is,

$$(x+1)^2 \;=\; x^2+2x+1.$$

By adding *and subtracting* the missing term in the given polynomial, we can put this perfect square in that polynomial:

$$x^2+2x-3 \;=\; x^2+2x+1-1-3 \;=\; \left(x^2+2x+1\right)-4 \;=\; (x+1)^2-4.$$

This process is called *completing the square.* Our next step is to *factor* this polynomial, or express it as a *product* of polynomials. To do this, we use the *Difference of Squares Formula:*

$$A^2-B^2 \;=\; (A-B)(A+B).$$

We can turn our polynomial into a "difference of squares" by replacing the 4 by 2^2; to use the Difference of Squares Formula, we let $A=x+1$ and $B=2$:

$$x^2+2x-3=(x+1)^2-4=(x+1)^2-2^2=((x+1)-2)((x+1)+2)=(x-1)(x+3).$$

Now, to find the roots of this polynomial, we use the fact that the product of two numbers is zero if and only if one or both of those numbers is zero:

$$x^2+2x-3=0 \implies (x-1)(x+3)=0 \implies (x-1=0 \text{ or } x+3=0) \implies (x=1 \text{ or } x=-3).$$

So, the roots are $\boxed{1 \text{ and } -3.}$

(b) $x^2+x-1=0$

Solution:

Again, we complete the square: $2k=1$, so $k=1/2$. Then, $k^2=(1/2)^2=1/4$, so

$$x^2+x-1 \;=\; x^2+x-1 \;=\; x^2+x+\frac{1}{4}-\frac{1}{4}-1 \;=\; \left(x^2+x+\frac{1}{4}\right)-\frac{1}{4}-\frac{4}{4}$$

$$= \left(x+\frac{1}{2}\right)^2-\frac{5}{4} == \left(x+\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2 \;=\; \left[\left(x+\frac{1}{2}\right)-\frac{\sqrt{5}}{2}\right]\left[\left(x+\frac{1}{2}\right)+\frac{\sqrt{5}}{2}\right].$$

(continued)

(continued) Then, $x^2+x-1=0 \implies \left[x+\frac{1}{2}-\frac{\sqrt{5}}{2}\right]\left[x+\frac{1}{2}+\frac{\sqrt{5}}{2}\right]=0$

$$\implies x+\frac{1}{2}-\frac{\sqrt{5}}{2}=0 \quad \text{or} \quad x+\frac{1}{2}+\frac{\sqrt{5}}{2}=0$$

$$\implies x=-\frac{1}{2}+\frac{\sqrt{5}}{2}=\frac{-1+\sqrt{5}}{2} \quad \text{or} \quad x=-\frac{1}{2}-\frac{\sqrt{5}}{2}=\frac{-1-\sqrt{5}}{2}.$$

Thus, the roots of this polynomial are $\boxed{\frac{-1+\sqrt{5}}{2} \text{ and } \frac{-1-\sqrt{5}}{2},}$ or $\frac{-1\pm\sqrt{5}}{2}$.

(c) $2x^2+x-10=0$

Solution:

To use the same method that we used in the previous parts of this question, we would like the coefficient of x^2 to be 1…no problem! We just *factor out* the 2:

$$2x^2+x-10 \;=\; 2\left[x^2+\frac{1}{2}x-5\right].$$

Then, $2k=1/2$, so $k=(1/2)/2=1/4$, and $k^2=(1/4)^2=1/16$. Accordingly,

$$2\left[x^2+\frac{1}{2}x-5\right] \;=\; 2\left[x^2+\frac{1}{2}x+\frac{1}{16}-\frac{1}{16}-\frac{5}{1}\right] \;=\; 2\left[\left(x^2+\frac{1}{2}x+\frac{1}{16}\right)-\frac{1}{16}-\frac{80}{16}\right]$$

$$=\; 2\left[\left(x+\frac{1}{4}\right)^2-\frac{81}{16}\right] \;=\; 2\left[\left(x+\frac{1}{4}\right)^2-\left(\frac{9}{4}\right)^2\right] \;=\; 2\left[\left(x+\frac{1}{4}\right)-\frac{9}{4}\right]\left[\left(x+\frac{1}{4}\right)+\frac{9}{4}\right]$$

$$=\; 2\left[x-\frac{8}{4}\right]\left[x+\frac{10}{4}\right] \;=\; 2\,[x-2]\left[x+\frac{5}{2}\right].$$

By observation, we see that the roots of this polynomial are $\boxed{2 \text{ and } -\frac{5}{2}.}$

(d) $(x+2)(x+5)(x-1)=0$

Solution:

We now use the fact that the product of two *or more* numbers is zero if and only if at least one of those numbers is zero:

$$(x+2)(x+5)(x-1)=0 \implies (x+2=0 \text{ or } x+5=0 \text{ or } x-1=0)$$

$$\implies (x=-2 \text{ or } x=-5 \text{ or } x=1), \text{ so the roots are } \boxed{-2,\ -5,\ \text{and } 1.}$$

(e) $x^6+2x^3-3=0$ (Hint: Let $y=x^3$)

Solution:

$x^6+2x^3-3 = \left(x^3\right)^2+2x^3-3 = y^2+2y-3$...HEY!!! We factored this in part (a)!*

Let us use that work here: $y^2+2y-3 = (y-1)(y+3) = \left(x^3-1\right)\left(x^3+3\right)$.

Equating this to zero, we find that $x^3=1$ or $x^3=-3$, so the roots are $\boxed{1 \text{ and } -\sqrt[3]{3}.}$

(5) Factor x^4+x^2+1. (Hint: The fastest way for this is to complete the square by adding and subtracting x^2.)

Solution:

You read the hint: $x^4+x^2+1 = x^4+x^2+1+x^2-x^2 = x^4+2x^2+1-x^2\ldots$

Hmmm...now what? Well, the first three terms look like a perfect square, just a *different* perfect square than the one that we used previously:

$$x^4+x^2+1 = \left(x^4+2x^2+1\right)-x^2 = \left(x^2+1\right)^2-x^2 = \boxed{\left[\left(x^2+1\right)-x\right]\left[\left(x^2+1\right)+x\right].}$$

*Well, the variable was x instead of y, but that's just a matter of labelling.