MATH 1410
Solutions for Homework 10
Submitted Friday, April 12, 2013

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(1) Let

$$A = \begin{bmatrix} 1 & 4 \\ 0.5 & 0 \end{bmatrix}, \quad \vec{\mathbf{v}}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

(a) Evaluate $A\vec{\mathbf{v}}_1$ and $A\vec{\mathbf{v}}_2$.

Solution:

$$A\vec{\mathbf{v}}_1 = \left[\begin{array}{cc} 1 & 4 \\ \hline 0.5 & 0 \end{array}\right]\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} (1)(4)+(4)(1) \\ (0.5)(4)+(0)(1) \end{bmatrix} = \boxed{\begin{bmatrix} 8 \\ 2 \end{bmatrix}} = 2\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2\vec{\mathbf{v}}_1.$$

Notice that we obtained a scalar multiple of $\vec{\mathbf{v}}_1$. Neat, ey? The idea that multiplying a vector by a number can produce that same result as multiplying it by a matrix is so important that we have labels for such things.

Let A be an $n \times n$ matrix, $\vec{\mathbf{x}}$ be a *nonzero* column vector from $\mathbb{R}^n$, and λ (the Greek letter *lambda*) be a number. If $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$, then we call λ an *eigenvalue* of A and call $\vec{\mathbf{x}}$ an associated *eigenvector*.

Accordingly, in our work above, we showed that $\lambda_1 = 2$ is an eigenvalue of A and $\vec{\mathbf{v}}_1$ is a corresponding eigenvector. Next, $A\vec{\mathbf{v}}_2$

$$= \left[\begin{array}{cc} 1 & 4 \\ \hline 0.5 & 0 \end{array}\right]\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} (1)(2)+(4)(-1) \\ (0.5)(2)+(0)(-1) \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ 1 \end{bmatrix}} = -\begin{bmatrix} 2 \\ -1 \end{bmatrix} = -\vec{\mathbf{v}}_2,$$

so $\vec{\mathbf{v}}_2$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

(b) Evaluate $A^{10}\vec{\mathbf{v}}_1$ and $A^{10}\vec{\mathbf{v}}_2$.

Solution:

YIKES! This is going to take FOREVER! *Unless...*

$$A^2\vec{\mathbf{v}}_i = A\big(A\vec{\mathbf{v}}_i\big) = A\big(\lambda_i\vec{\mathbf{v}}_i\big) = \lambda_i\big(A\vec{\mathbf{v}}_i\big) = \lambda_i\big(\lambda_i\vec{\mathbf{v}}_i\big) = \lambda_i^2\vec{\mathbf{v}}_i.$$

(continued)

(continued) Then, $A^3\vec{\mathbf{v}}_i = A\big(A^2\vec{\mathbf{v}}_i\big) = A\big(\lambda_i^2\vec{\mathbf{v}}_i\big) = \lambda_i^2\big(A\vec{\mathbf{v}}_i\big) = \lambda_i^2\big(\lambda_i\vec{\mathbf{v}}_i\big) = \lambda_i^3\vec{\mathbf{v}}_i$.

Next, $A^4\vec{\mathbf{v}}_i = A\big(A^3\vec{\mathbf{v}}_i\big) = A\big(\lambda_i^3\vec{\mathbf{v}}_i\big) = \lambda_i^3\big(A\vec{\mathbf{v}}_i\big) = \lambda_i^3\big(\lambda_i\vec{\mathbf{v}}_i\big) = \lambda_i^4\vec{\mathbf{v}}_i$.

Repeating this *ad infinitum,* we find that for *any* positive integer n, $A^n\vec{\mathbf{v}}_i = \lambda_i^n\vec{\mathbf{v}}_i$. In general, if $\vec{\mathbf{x}}$ is an eigenvector of A associated with the eigenvalue λ and n is any positive integer, then

$$A^n\vec{\mathbf{x}} = \lambda^n\vec{\mathbf{x}}.$$

In our case, $\vec{\mathbf{v}}_1$ corresponds to the eigenvalue $\lambda_1 = 2$, so we have

$$A^{10}\vec{\mathbf{v}}_1 = \lambda_1^{10}\vec{\mathbf{v}}_1 = 2^{10}\vec{\mathbf{v}}_1 = 1024\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4096 \\ 1024 \end{bmatrix}.}$$

Since $\vec{\mathbf{v}}_2$ is associated with the eigenvalue $\lambda_2 = -1$, we also have

$$A^{10}\vec{\mathbf{v}}_2 = \lambda_2^{10}\vec{\mathbf{v}}_2 = (-1)^{10}\vec{\mathbf{v}}_2 = (+1)\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -1 \end{bmatrix}.}$$

(c) Find c_1 and c_2 so that

$$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 = \vec{\mathbf{w}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

<u>Solution</u>:

So, Linear Combination, at last we meet again for the first time for the last time!

$$\left[\begin{array}{cc|c} 4 & 2 & 2 \\ 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 4 & 2 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 - 4R_1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 6 & -2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{6}R_2} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & -1/3 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2} \left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & -1/3 \end{array}\right].$$

So, $c_1 = \boxed{2/3}$ and $c_2 = \boxed{-1/3;}$ that is, $\vec{\mathbf{w}} = (2/3)\vec{\mathbf{v}}_1 - (1/3)\vec{\mathbf{v}}_2$.

(d) Evaluate $A^{10}\vec{\mathbf{w}}$.

Solution:

$$A^{10}\vec{\mathbf{w}} \;=\; A^{10}\left(\frac{2}{3}\vec{\mathbf{v}}_1 - \frac{1}{3}\vec{\mathbf{v}}_2\right) \;=\; \frac{2}{3}\left(A^{10}\vec{\mathbf{v}}_1\right) - \frac{1}{3}\left(A^{10}\vec{\mathbf{v}}_2\right)$$

$$= \frac{2}{3}\begin{bmatrix} 4096 \\ 1024 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8192/3 \\ 2048/3 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 8190/3 \\ 2049/3 \end{bmatrix} = \boxed{\begin{bmatrix} 2730 \\ 683 \end{bmatrix}}.$$

(e) Evaluate $A^{10}\vec{\mathbf{e}}_1$ and $A^{10}\vec{\mathbf{e}}_2$. (As usual, $\vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.)

Solution:

Hmmm…to use the same shortcut that we used in (d), we need to express *both* $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$ as linear combinations of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$. I know! Let's use multiple solution columns:

$$\left[\begin{array}{rr|rr} 4 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rr|rr} 1 & -1 & 0 & 1 \\ 4 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - 4R_1} \left[\begin{array}{rr|rr} 1 & -1 & 0 & 1 \\ 0 & 6 & 1 & -4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{6}R_2} \left[\begin{array}{rr|rr} 1 & -1 & 0 & 1 \\ 0 & 1 & 1/6 & -2/3 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2} \left[\begin{array}{rr|rr} 1 & 0 & 1/6 & 1/3 \\ 0 & 1 & 1/6 & -2/3 \end{array}\right].$$

Finding each sequence of coefficients in the appropriate solution column, we get

$$\vec{\mathbf{e}}_1 \;=\; (1/6)\vec{\mathbf{v}}_1 + (1/6)\vec{\mathbf{v}}_2 \quad \text{and} \quad \vec{\mathbf{e}}_2 \;=\; (1/3)\vec{\mathbf{v}}_1 - (2/3)\vec{\mathbf{v}}_2\,.$$

Then, $A^{10}\vec{\mathbf{e}}_1 \;=\; A^{10}\left(\frac{1}{6}\vec{\mathbf{v}}_1 + \frac{1}{6}\vec{\mathbf{v}}_2\right) \;=\; \frac{1}{6}\left(A^{10}\vec{\mathbf{v}}_1\right) + \frac{1}{6}\left(A^{10}\vec{\mathbf{v}}_2\right)$

$$= \frac{1}{6}\begin{bmatrix} 4096 \\ 1024 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 4098 \\ 1023 \end{bmatrix} = \boxed{\begin{bmatrix} 683 \\ 170.5 \end{bmatrix}}.$$

(continued)

(continued) Next, $A^{10}\overrightarrow{\mathbf{e}}_2 = A^{10}\left(\frac{1}{3}\overrightarrow{\mathbf{v}}_1 - \frac{2}{3}\overrightarrow{\mathbf{v}}_2\right) = \frac{1}{3}\left(A^{10}\overrightarrow{\mathbf{v}}_1\right) - \frac{2}{3}\left(A^{10}\overrightarrow{\mathbf{v}}_2\right)$

$$= \frac{1}{3}\begin{bmatrix} 4096 \\ 1024 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4096/3 \\ 1024/3 \end{bmatrix} - \begin{bmatrix} 4/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 4092/3 \\ 1026/3 \end{bmatrix} = \boxed{\begin{bmatrix} 1364 \\ 342 \end{bmatrix}}.$$

(f) Evaluate A^{10}.

Solution:

$$A^{10} = A^{10} I_2 = A^{10}\left[\ \overrightarrow{\mathbf{e}}_1 \mid \overrightarrow{\mathbf{e}}_2\ \right] = \left[\ A^{10}\overrightarrow{\mathbf{e}}_1 \mid A^{10}\overrightarrow{\mathbf{e}}_2\ \right] = \boxed{\begin{bmatrix} 683 & 1364 \\ 170.5 & 342 \end{bmatrix}}.$$

(2) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \overrightarrow{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \overrightarrow{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \overrightarrow{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

(a) Calculate $A\overrightarrow{\mathbf{v}}_1$, $A\overrightarrow{\mathbf{v}}_2$, and $A\overrightarrow{\mathbf{v}}_3$.

Solution:

First, $$A\overrightarrow{\mathbf{v}}_1 = \begin{bmatrix} 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2\overrightarrow{\mathbf{v}}_1.$$

We see that $\overrightarrow{\mathbf{v}}_1$ is an eigenvector of A that corresponds to the eigenvalue $\lambda_1 = 2$. Next,

$$A\overrightarrow{\mathbf{v}}_2 = \begin{bmatrix} 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} = \overrightarrow{\mathbf{v}}_2 = 1\overrightarrow{\mathbf{v}}_2,$$

so $\overrightarrow{\mathbf{v}}_2$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 1$.

(continued)

(continued) Lastly, $A\overrightarrow{\mathbf{v}}_3 = \left[\begin{array}{ccc} 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 1 \end{array}\right] \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}} = -\overrightarrow{\mathbf{v}}_3$,

so $\overrightarrow{\mathbf{v}}_3$ is an eigenvector of A that is associated with the eigenvalue $\lambda_3 = -1$.

(b) Calculate $A^k\,\overrightarrow{\mathbf{v}}_1$, $A^k\,\overrightarrow{\mathbf{v}}_2$, and $A^k\,\overrightarrow{\mathbf{v}}_3$.

Solution:

Again, we use the formula $A^k\,\overrightarrow{\mathbf{x}} = \lambda^k\,\overrightarrow{\mathbf{x}}$ where $\overrightarrow{\mathbf{x}}$ is an eigenvector of A associated with the eigenvalue λ and k is any positive integer:

$$A^k\,\overrightarrow{\mathbf{v}}_1 = \lambda_1^k\,\overrightarrow{\mathbf{v}}_1 = 2^k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2^k \\ 2^k \\ 2^k \end{bmatrix}}.$$

$$A^k\,\overrightarrow{\mathbf{v}}_2 = \lambda_2^k\,\overrightarrow{\mathbf{v}}_2 = 1^k\,\overrightarrow{\mathbf{v}}_2 = \overrightarrow{\mathbf{v}}_2 = \boxed{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}.$$

$$A^k\,\overrightarrow{\mathbf{v}}_3 = \lambda_3^k\,\overrightarrow{\mathbf{v}}_3 = (-1)^k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} (-1)^k \\ -2(-1)^k \\ (-1)^k \end{bmatrix}}.$$

(c) Find c_1, c_2, and c_3 such that

$$c_1 \overrightarrow{\mathbf{v}}_1 + c_2 \overrightarrow{\mathbf{v}}_2 + c_3 \overrightarrow{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

<u>Solution</u>:

You know the drill!

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 2 \\ 1 & -1 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} R_2 - R_1 \\ R_3 - R_1 \\ \longrightarrow \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & -1 & -3 & 1 \\ 0 & -2 & 0 & 2 \end{array}\right]$$

$$\begin{array}{c} -R_2 \\ \longrightarrow \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & -2 & 0 & 2 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ R_3 + 2R_2 \\ \longrightarrow \end{array} \left[\begin{array}{rrr|r} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 6 & 0 \end{array}\right]$$

$$\begin{array}{c} \frac{1}{6} R_3 \\ \longrightarrow \end{array} \left[\begin{array}{rrr|r} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 + 2R_3 \\ R_2 - 3R_3 \\ \longrightarrow \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Thus, $c_1 = \boxed{2,}$ $c_2 = \boxed{-1,}$ and $c_3 = \boxed{0.}$

In other words, $2\overrightarrow{\mathbf{v}}_1 - \overrightarrow{\mathbf{v}}_2 + 0\overrightarrow{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2\overrightarrow{\mathbf{v}}_1 - \overrightarrow{\mathbf{v}}_2.$

(d) Calculate $A^k \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Solution:

$$A^k \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = A^k\left(2\overrightarrow{\mathbf{v}}_1 - \overrightarrow{\mathbf{v}}_2\right) = 2\left(A^k\overrightarrow{\mathbf{v}}_1\right) - A^k\overrightarrow{\mathbf{v}}_2 = 2\begin{bmatrix} 2^k \\ 2^k \\ 2^k \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\left(2^k\right) \\ 2\left(2^k\right) \\ 2\left(2^k\right) \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2\left(2^k\right)-1 \\ 2\left(2^k\right) \\ 2\left(2^k\right)+1 \end{bmatrix}} = \begin{bmatrix} 2^{k+1}-1 \\ 2^{k+1} \\ 2^{k+1}+1 \end{bmatrix}.$$

(3) Let $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$.

(a) Find nonzero vectors $\overrightarrow{\mathbf{v}}$ and $\overrightarrow{\mathbf{w}}$ so that $(A-3I_2)\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{0}}$ and $(A+2I_2)\overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{0}}$.

Solution:

$$(A-3I_2)\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{0}} \implies \left(\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{0}}$$

$$\implies \left(\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{0}} \implies \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{0}}.$$

This is the matrix equation form† of a system of linear equations whose corresponding augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 4 & 0 \\ 1 & -4 & 0 \end{array}\right].$$

(continued)

†The *matrix equation form* of a system of linear equations is $A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{b}}$, where A is the *coefficient matrix* (obtained by deleting the solution column in the associated augmented matrix), $\overrightarrow{\mathbf{x}}$ is the *variable (column) vector,* which contains all the variables in the system, and $\overrightarrow{\mathbf{b}}$ is the *constant (column) vector,* which contains all the constants in the system (in other words, $\overrightarrow{\mathbf{b}}$ is the solution column in the corresponding augmented matrix). Note that the augmented matrix for this system is $[\,A \mid \overrightarrow{\mathbf{b}}\,]$.

(continued) We can now solve the system by row reducing this matrix:

$$\left[\begin{array}{cc|c} -1 & 4 & 0 \\ 1 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{-R_1} \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 1 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} \textcircled{1} & -4 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Say that the variables are x and y. Then, y is free, so we let $y = t$. Next,

$$x - 4y = 0 \implies x = 4y = 4t.$$

So, the system has infinitely many solutions of the form $x = 4t,\ y = t$. To ensure that our vector $\overrightarrow{\mathbf{v}}$ is *nonzero,* let us choose one nonzero solution by selecting a nonzero value for t

$$\text{i.e.}\quad t = 1 \implies (\ x = 4(1) = 4 \ \text{ and } \ y = 1\ ) \implies \overrightarrow{\mathbf{v}} = \boxed{\begin{bmatrix} 4 \\ 1 \end{bmatrix}.}$$

$$\text{Next,}\quad (A + 2I_2)\overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{0}} \implies \left(\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{0}}$$

$$\implies \left(\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{0}} \implies \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{0}}.$$

This is the matrix equation form of a system whose associated augmented matrix is

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - 4R_1} \left[\begin{array}{cc|c} \textcircled{1} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Let the variables be x and y. Then, y is free, so $y = t$, and

$$x + y = 0 \implies x = -y = -t.$$

The system therefore has infinitely many solutions of the form $x = -t,\ y = t$. To create $\overrightarrow{\mathbf{w}}$, we need a single nonzero solution, and we may get it by choosing a nonzero value for t:

$$t = 1 \implies (\ x = -1 \ \text{ and } \ y = 1\ ) \implies \overrightarrow{\mathbf{w}} = \boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}.}$$

(b) Show that $A\vec{\mathbf{v}} = 3\vec{\mathbf{v}}$ and $A\vec{\mathbf{w}} = -2\vec{\mathbf{w}}$.

Solution:

$$A\vec{\mathbf{v}} = \left[\begin{array}{cc} 2 & 4 \\ \hline 1 & -1 \end{array}\right]\left[\begin{array}{c} 4 \\ 1 \end{array}\right] = \left[\begin{array}{c} (2)(4)+(4)(1) \\ (1)(4)+(-1)(1) \end{array}\right] = \left[\begin{array}{c} 12 \\ 3 \end{array}\right] = 3\left[\begin{array}{c} 4 \\ 1 \end{array}\right] = 3\vec{\mathbf{v}},$$

as required. Next,

$$A\vec{\mathbf{w}} = \left[\begin{array}{cc} 2 & 4 \\ \hline 1 & -1 \end{array}\right]\left[\begin{array}{c} -1 \\ 1 \end{array}\right] = \left[\begin{array}{c} 2 \\ -2 \end{array}\right] = -2\left[\begin{array}{c} -1 \\ 1 \end{array}\right] = -2\vec{\mathbf{w}}, \text{ as required.}$$

Why did these work? Consider this:

$$A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}} \iff A\vec{\mathbf{x}} - \lambda\vec{\mathbf{x}} = \vec{\mathbf{0}} \iff A\vec{\mathbf{x}} - \lambda I_n\vec{\mathbf{x}} = \vec{\mathbf{0}} \iff (A - \lambda I_n)\vec{\mathbf{x}} = \vec{\mathbf{0}}.$$

(4) Let $A = \left[\begin{array}{cc} 1 & 3 \\ 4 & 2 \end{array}\right]$.

(a) Find [the] eigenvalues of A i.e. find real numbers λ_1 and λ_2 such that $(A - \lambda_i I_2)$ is not invertible.

Solution:

Let us first construct the matrix $A - \lambda I_2$

$$= \left[\begin{array}{cc} 1 & 3 \\ 4 & 2 \end{array}\right] - \lambda\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] = \left[\begin{array}{cc} 1 & 3 \\ 4 & 2 \end{array}\right] - \left[\begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array}\right] = \left[\begin{array}{cc} 1-\lambda & 3 \\ 4 & 2-\lambda \end{array}\right].$$

The *determinant* of a square matrix A, denoted $\det(A)$ or $|A|$, is one number associated with the entire matrix that is *nonzero* if and only if the matrix is *invertible.* For 2×2 matrices, the determinant is defined as

$$\left|\begin{array}{cc} a & b \\ c & d \end{array}\right| = ad - bc.$$

Accordingly, $\det(A - \lambda I_2) = \left|\begin{array}{cc} 1-\lambda & 3 \\ 4 & 2-\lambda \end{array}\right| = (1-\lambda)(2-\lambda) - (3)(4)$

$$= 2 - \lambda - 2\lambda + \lambda^2 - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2).$$

Notice that this a *polynomial* whose variable is λ instead of x; we call it the *characteristic polynomial of* A. Since we want $A - \lambda I_2$ to *not* be invertible, we want its determinant to be *zero;* in other words, the eigenvalues of A are the *roots* of its characteristic polynomial:

$$\lambda_1 = \boxed{5} \quad \text{and} \quad \lambda_2 = \boxed{-2.}$$

(b) Find [the] eigenvectors of A i.e. find nonzero vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ such that $A\vec{\mathbf{v}}_i = \lambda_i \vec{\mathbf{v}}_i$.

Solution:

As we saw in (3), we solve $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ by solving $(A - \lambda I_n)\,\vec{\mathbf{x}} = \vec{\mathbf{0}}$, which in turn is done by creating and row reducing the augmented matrix $\left[\; A - \lambda I_n \mid \vec{\mathbf{0}} \;\right]$. While working on (4)(a), we may also have realized that $A - \lambda I_n$ can be created quickly by subtracting λ from each entry on the main diagonal of A!

Since we have two eigenvalues, we have to solve *two* systems of linear equations:

λ_1 = 5:

$$\left[\; A - 5I_n \mid \vec{\mathbf{0}} \;\right] = \left[\begin{array}{cc|c} 1-5 & 3 & 0 \\ 4 & 2-5 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 4 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + R_1} \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{4}R_1} \left[\begin{array}{cc|c} \textcircled{1} & -3/4 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right].$$

Notice that this RREF has a row of zeros: this should *always* happen when solving for eigenvectors! If it does not, then we have made a mistake somewhere. If we did not make any arithmetic errors while row reducing, then the mistake must be the value of λ: if it is *not* an eigenvalue, then we will *not* get a row of zeros.

Anyways, the system has infinitely many solutions of the form $x = (3/4)\,t,\ y = t$. When choosing our nonzero value for t, we can select one that will eliminate any fractions in the vector (YAY!):

$$t = 4 \implies (\ x = (3/4)(4) = 3 \ \text{ and } \ y = 4\ ) \implies \vec{\mathbf{v}}_1 = \boxed{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}.$$

(continued)

(continued) **$\lambda_2 = -2$:** $\left[\begin{array}{c|c} A-(-2)I_n & \overrightarrow{\mathbf{0}} \end{array}\right] = \left[\begin{array}{c|c} A+2I_n & \overrightarrow{\mathbf{0}} \end{array}\right]$

$$= \left[\begin{array}{cc|c} 1+2 & 3 & 0 \\ 4 & 2+2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2-4R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right].$$

As expected, this RREF has a row of zeros. The system has infinitely many solutions of the form $x = -t,\ y = t$, so we let

$$t = 1 \implies (\ x = -1 \ \text{ and } \ y = 1\ ) \implies \overrightarrow{\mathbf{v}}_2 = \boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}.}$$

(c) Let $S = \begin{bmatrix} \overrightarrow{\mathbf{v}}_1 & \overrightarrow{\mathbf{v}}_2 \end{bmatrix}$. Calculate S^{-1}.

Solution:

To begin, $S = \begin{bmatrix} \overrightarrow{\mathbf{v}}_1 & \overrightarrow{\mathbf{v}}_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix}$.[‡]

To find the inverse of an $n \times n$ matrix A, we use the *Matrix Inversion Algorithm:* we create and row reduce the $n \times (2n)$ matrix $[\ A \mid I_n\ ]$ until we get its reduced row echelon form $[\ R \mid B\ ]$; if $R = I_n$, then A is invertible and $A^{-1} = B$, but if $R \neq I_n$ (that is, it has a row of zeros), then A is not invertible.

(continued)

[‡]Note that we could have chosen -2 as our first eigenvalue and 5 as our second, in which case the columns in S would be swapped. That's okay! There are *multiple* correct answers.

(continued) Let us apply this algorithm to S:

$$
\begin{array}{rcl}
[\, S \mid I_2 \,] & = & \left[\begin{array}{rr|rr} 3 & -1 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right] \\[2ex]
& \xrightarrow{R_2 - R_1} & \left[\begin{array}{rr|rr} 3 & -1 & 1 & 0 \\ 1 & 2 & -1 & 1 \end{array}\right] \\[2ex]
& \xrightarrow{R_1 \leftrightarrow R_2} & \left[\begin{array}{rr|rr} 1 & 2 & -1 & 1 \\ 3 & -1 & 1 & 0 \end{array}\right] \\[2ex]
& \xrightarrow{R_2 - 3R_1} & \left[\begin{array}{rr|rr} 1 & 2 & -1 & 1 \\ 0 & -7 & 4 & -3 \end{array}\right] \\[2ex]
& \xrightarrow{-\frac{1}{7}R_2} & \left[\begin{array}{rr|rr} 1 & 2 & -7/7 & 7/7 \\ 0 & 1 & -4/7 & 3/7 \end{array}\right] \\[2ex]
& \xrightarrow{R_1 - 2R_2} & \left[\begin{array}{rr|rr} 1 & 0 & 1/7 & 1/7 \\ 0 & 1 & -4/7 & 3/7 \end{array}\right].
\end{array}
$$

We see I_2 to the left of the partition, so $S^{-1} = \boxed{\begin{bmatrix} 1/7 & 1/7 \\ -4/7 & 3/7 \end{bmatrix}.}$

(d) Calculate $M = S^{-1}AS$.

<u>Solution</u>:

$$
M = S^{-1}(AS) = S^{-1}\left(\left[\begin{array}{rr} 1 & 3 \\ \hline 4 & 2 \end{array}\right]\left[\begin{array}{r|r} 3 & -1 \\ 4 & 1 \end{array}\right]\right) = S^{-1}\left(\begin{bmatrix} 15 & 2 \\ 20 & -2 \end{bmatrix}\right)
$$

$$
= \left[\begin{array}{rr} 1/7 & 1/7 \\ \hline -4/7 & 3/7 \end{array}\right]\left[\begin{array}{r|r} 15 & 2 \\ 20 & -2 \end{array}\right] = \begin{bmatrix} 35/7 & 0/7 \\ 0/7 & -14/7 \end{bmatrix} = \boxed{\begin{bmatrix} \mathbf{5} & 0 \\ 0 & -\mathbf{2} \end{bmatrix}.}
$$

Notice that the entries on the main diagonal of M are the *eigenvalues* of A. Moreover, the number of the column that each eigenvalue is in *matches* the number of the column in S in which we put its corresponding eigenvector. This is *not* a coinkydink! If we had swapped the columns in S before finding its inverse and computing $S^{-1}AS$, we would have gotten

$$
M = \boxed{\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}.}
$$

(5) Calculate det(A) where

$$A = \begin{bmatrix} 2 & -4 & 5 \\ 0 & -3 & 6 \\ 4 & 5 & 7 \end{bmatrix}.$$

Solution:

To calculate the determinant of an $n \times n$ matrix A where n exceeds 2, we use *Laplace's cofactor expansion:* we pick any row or column in A, then move along that row or column multiplying each (i, j)-entry that we encounter by $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ "submatrix" obtained by removing row i and column j from A, then sum the results.

To illustrate that we can pick *any* row or column and still achieve the same result, let us compute the determinant twice, first using row 2 and then using column 1:

Row 2:

$$\det(A) = \begin{vmatrix} 2 & -4 & 5 \\ \mathbf{0} & \mathbf{-3} & \mathbf{6} \\ 4 & 5 & 7 \end{vmatrix} = \mathbf{0} - \mathbf{3}(-1)^{2+2} \begin{vmatrix} 2 & 5 \\ 4 & 7 \end{vmatrix} + \mathbf{6}(-1)^{2+3} \begin{vmatrix} 2 & -4 \\ 4 & 5 \end{vmatrix}$$

$$= -3(-1)^4[(2)(7)-(5)(4)] + 6(-1)^5[(2)(5)-(-4)(4)]$$

$$= -3(+1)[14-20] + 6(-1)[10+16] = -3[-6] - 6[26] = 18 - 156 = \boxed{-138.}$$

Column 1:

$$\det(A) = \begin{vmatrix} \mathbf{2} & -4 & 5 \\ \mathbf{0} & -3 & 6 \\ \mathbf{4} & 5 & 7 \end{vmatrix} = \mathbf{2}(-1)^{1+1} \begin{vmatrix} -3 & 6 \\ 5 & 7 \end{vmatrix} + \mathbf{0} + \mathbf{4}(-1)^{3+1} \begin{vmatrix} -4 & 5 \\ -3 & 6 \end{vmatrix}$$

$$= 2(-1)^2[(-3)(7)-(6)(5)] + 4(-1)^4[(-4)(6)-(5)(-3)]$$

$$= 2(+1)[-21-30] + 4(+1)[-24+15] = 2[-51] + 4[-9] = -102 - 36 = \boxed{-138.}$$