MATH 1410
Solutions for Homework 2
Submitted Friday, January 25

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(1) Find all solutions to the following systems of equations:

(a) $$\begin{cases} 2x + 3y + 4z = 1 \\ 4x - 9y + 16z = 1 \\ 3x + 3y - z = 2. \end{cases}$$

<u>Solution</u>:

Substitution was used more often in the solutions for Homework 1, so let us solve these systems using elimination. To begin, we number the equations:

$$\begin{cases} ①: \ 2x + 3y + 4z = 1 \\ ②: \ 4x - 9y + 16z = 1 \\ ③: \ 3x + 3y - z = 2. \end{cases}$$

Let us now eliminate the x. ② has $4x$ in it, while ① has $2x$ in it, so if we multiply both sides of ① by 2 and subtract the resulting equation from ②[†], the $4x$'s will cancel and the x will be eliminated:

$$② - 2 \times ①: \ 4x - 9y + 16z - 2(2x + 3y + 4z) = 1 - 2(1)$$

$$\implies 4x - 9y + 16z - 4x - 6y - 8z = 1 - 2 \implies ④: \ -15y + 8z = -1.$$

To eliminate the x in ③, we multiply both sides of that equation by 2 (to turn the $3x$ into a $6x$), multiply both sides of ① by 3 (to turn the $2x$ into a $6x$), and then subtract the resulting equations:

$$2 \times ③ - 3 \times ①: \ 2(3x + 3y - z) - 3(2x + 3y + 4z) = 2(2) - 3(1)$$

$$\implies 6x + 6y - 2z - 6x - 9y - 12z = 4 - 3 \implies ⑤: \ -3y - 14z = 1.$$

Notice that ④ and ⑤ form their *own* system of two equations in two unknowns. We now need to solve this system.

(continued)

[†]To "subtract equations," we equate the difference in their left-hand sides to the difference in their right-hand sides (using the same order in each difference, of course). Similarly, we "add equations" by equating the sum of their left-hand sides to the sum of their right-hand sides.

(continued) To solve this new system, we eliminate the y by multiplying both sides of ⑤ by 5 and subtracting the resulting equation from ④:

$$④ - 5 \times ⑤: \quad -15y + 8z - 5(-3y - 14z) = -1 - 5(1)$$

$$\Longrightarrow \quad -15y + 8z + 15y + 70z = -1 - 5 \quad \Longrightarrow \quad 78z = -6 \quad \Longrightarrow \quad z = -\frac{6}{78} = -\frac{6}{6 \cdot 13} = -\frac{1}{13}.$$

We now "back-substitute." To begin, we substitute our z-value into an equation whose variables are just y and z (like ⑤) to get a value for y:

$$⑤: \ -3y - 14z = 1 \quad \Longrightarrow \quad -3y - 14\left(-\frac{1}{13}\right) = \frac{1}{1} \quad \Longrightarrow \quad -3y + \frac{14}{13} = \frac{13}{13}$$

$$\Longrightarrow \quad -3y = \frac{13}{13} - \frac{14}{13} \quad \Longrightarrow \quad -3y = -\frac{1}{13} \quad \Longrightarrow \quad y = -\frac{1}{3}\left(-\frac{1}{13}\right) = \frac{1}{39}.$$

Finally, we subsitute our known values for y and z into an equation whose variables are x, y, and z (like ③) to get a value for x:

$$③: \ 3x + 3y - z = 2 \quad \Longrightarrow \quad 3x = 2 - 3y + z$$

$$\Longrightarrow \quad x = \frac{2}{3} - y + \frac{1}{3}z = \frac{2}{3} - \left(\frac{1}{39}\right) + \frac{1}{3}\left(-\frac{1}{13}\right) = \frac{26}{39} - \frac{1}{39} - \frac{1}{39} = \frac{24}{39} = \frac{8}{13}.$$

Hence, this system has one solution: $(x, y, z) = \boxed{\left(\frac{8}{13}, \frac{1}{39}, -\frac{1}{13}\right).}$

(b) $\begin{cases} x + y = 1 \\ 2x + 3y = 0 \\ 4x + 9y = -1. \end{cases}$

Solution:

As usual, we number the equations: $\begin{cases} ①: \ x + y = 1 \\ ②: \ 2x + 3y = 0 \\ ③: \ 4x + 9y = -1. \end{cases}$

We first eliminate the x:

$$②-2\times①: \ 2x+3y-2(x+y) = 0-2(1) \quad \Longrightarrow \quad 2x+3y-2x-2y = 0-2 \quad \Longrightarrow \quad ④: \ y = -2.$$

(continued)

(continued) Next, $③ - 4 \times ①:\ 4x + 9y - 4(x+y) = -1 - 4(1)$

$\implies\ 4x + 9y - 4x - 4y = -1 - 4 \implies 5y = -5 \implies ⑤:\ y = -1\ldots$

WAITAMINUTE! According to ⑤, y must be -1, but according to ④, y must *also* be -2! There is no value that we can assign to y that will satisfy both equations simultaneously! (O_o)

We therefore conclude that the given system has $\boxed{\textit{no}\text{ solution.}}$

(c) $\begin{cases} a + b + c = 2 \\ 2a - 2b + c = 5 \\ 3a - b + 2c = 7. \end{cases}$

Solution:

As per usual: $\begin{cases} ①:\ a + b + c = 2 \\ ②:\ 2a - 2b + c = 5 \\ ③:\ 3a - b + 2c = 7. \end{cases}$

Let's eliminate the a:

$② - 2 \times ①:\ 2a - 2b + c - 2(a+b+c) = 5 - 2(2)$

$\implies\ 2a - 2b + c - 2a - 2b - 2c = 5 - 4 \implies ④:\ -4b - c = 1.$

$③ - 3 \times ①:\ 3a - b + 2c - 3(a+b+c) = 7 - 3(2)$

$\implies\ 3a - b + 2c - 3a - 3b - 3c = 7 - 6 \implies ⑤:\ -4b - c = 1\ldots$

WAITAMINUTE! ④ and ⑤ are the SAME equation! (O_o)

Clearly, one of them is redundant, which leaves us with *one* equation. This is not sufficient to give us fixed values for *two* variables; instead, one variable is *free* to take any value, while the other is *dependent* on our choice. For example, I can make c equal to 3, but that would force b to be -1; similarly, b can be 2, but only if c is -9.

Which one do we make free and which one becomes dependent? Either choice is fine. On the other hand, solving ⑤ for c in terms of b is "nicer" than solving it for b in terms of c (the latter introduces fractions... yuck!), so let's make b the free variable.

Specifically, let's pick some number, t, and set b equal to it i.e. $b = t$. Then,

$$⑤:\ -4b - c = 1 \implies -4t - c = 1 \implies -c = 1 + 4t \implies c = -1 - 4t.$$

(continued)

(continued) Then,

$$\text{①}: \; a+b+c=2 \implies a+t+(-1-4t)=2 \implies a-1-3t=2 \implies a=3+3t.$$

Because there are *infinitely many* choices for the number t, the given system has infinitely many solutions of the form

$$(a, b, c) = \boxed{(3+3t, t, -1-4t).}$$

Using *vector algebra*, it is possible to "decompose" this solution as follows:

$$(3+3t, t, -1-4t) = (3, 0, -1) + (3t, t, -4t) = (3, 0, -1) + t(3, 1, -4).$$

Soon in the course, we will see that this decomposition allows us to interpret the solution *geometrically*.

Say we had made c the free variable, instead; that is, $c = t$. Then,

$$\text{⑤}: \; -4b-c=1 \implies -4b-t=1 \implies -4b=1+t \implies b=-\frac{1}{4}-\frac{1}{4}t.$$

Then, $\text{①}: \; a+b+c=2 \implies a+\left(-\frac{1}{4}-\frac{1}{4}t\right)+t=2$

$$\implies a = 2+\frac{1}{4}+\frac{1}{4}t-t = \frac{8}{4}+\frac{1}{4}+\frac{1}{4}t-\frac{4}{4}t = \frac{9}{4}-\frac{3}{4}t.$$

In this case, the given system has infinitely many solutions of the form

$$(a, b, c) = \boxed{\left(\frac{9}{4}-\frac{3}{4}t, -\frac{1}{4}-\frac{1}{4}t, t\right)}$$

$$= \left(\frac{9}{4}, -\frac{1}{4}, 0\right) + \left(-\frac{3}{4}t, -\frac{1}{4}t, t\right) = \left(\frac{9}{4}, -\frac{1}{4}, 0\right) + t\left(\frac{3}{4}, -\frac{1}{4}, 1\right).$$

(2) Find three equations and three unknowns such that $(x, y, z) = (2, 4, 1)$ [are] the only solutions.

<u>Solution</u>:

What does $(x, y, z) = (2, 4, 1)$ mean, anyway? It means $x = 2$, $y = 4$, and $z = 1$. Which is three equations. With three variables. Which is what the question wanted. So, one possible answer is

$$\begin{cases} x & & & = & 2 \\ & y & & = & 4 \\ & & z & = & 1. \end{cases}$$

Bada bing, bada boom! (^‿^)

To get a more...well, *sophisticated* answer, we create three linear expressions: one that contains only the variable z, another that contains only y and z, and one more that contains only x, y, and z, such as these:

$$2x - 3y + 4z, \;\; 5y - 6z, \;\; \text{and} \;\; 7z.$$

We then substitute the values from our desired solution into these expressions:

$$2x - 3y + 4z \;=\; 2(2) - 3(4) + 4(1) \;=\; 4 - 12 + 4 \;=\; -4,$$

$$5y - 6z \;=\; 5(4) - 6(1) \;=\; 20 - 6 \;=\; 14, \;\; \text{and} \;\; 7z \;=\; 7(1) \;=\; 7.$$

Equating each of the original expressions to its output when the solution was substituted into it yields this system:

$$\begin{cases} 2x & - & 3y & + & 4z & = & -4 \\ & & 5y & - & 6z & = & 14 \\ & & & & 7z & = & 7. \end{cases}$$

Does this work? Well, by our choice of the right-hand side of each equation, we know that $(x, y, z) = (2, 4, 1)$ *is* a solution. How do we know that there aren't *more* solutions? Well, according to the third equation, z *must* be 1. Then, according to the second equation and the fact that $z = 1$, y *must* be 4. Finally, according to the first equation and the facts that $y = 4$ and $z = 1$, x *must* be 2. Hence, this system is an acceptable answer to the question.

It is possible to create even more sophisticated examples. We will see how when we discuss Gaussian Elimination later in the course.

(3) Find two equations and two [unknowns] such that $(x, y) = (0, 2) + t(1, 1)$ are all the solutions to this system.

Solution:

The given solution has been decomposed, so let us reconstruct it:

$$(x, y) \;=\; (0, 2) + t(1, 1) \;=\; (0, 2) + (t, t) \;=\; (t, 2+t);$$

that is, $x = t$ and $y = 2 + t$. We may have two equations, but we also have *three* unknowns (x, y, *and* t). One of these unknowns needs to have. . . an unfortunate accident, heh, heh, heh! ($\substack{\otimes\\ \otimes}$P) Did I just say that out loud? I meant, we need to replace one of them. Since $x = t$, we can substitute this into the other equation:

$$y = 2 + t \implies y = 2 + x \implies -x + y = 2.$$

Now we need a second equation; one whose inclusion won't change the solutions, like. . . a multiple! If we multiply both sides of the equation by a nonzero number (say, -2), we will get a *different* equation that preserves the relationship between x and y:

$$-2(-x+y) = -2(2) \implies -2(-x) + (-2)y = -4 \implies 2x - 2y = -4.$$

So, a system with two equations and two unknowns with the desired solution set is

$$\boxed{\left\{ \begin{array}{rcrcr} -x & + & y & = & 2 \\ 2x & - & 2y & = & -4. \end{array} \right.}$$

(4) Let $\overrightarrow{OA} = [3, 4]$ and $\overrightarrow{OB} = [2, 2]$. Evaluate $\overrightarrow{AB}$.

Solution:

A *vector* is, essentially, a change in position. For instance, if I'm standing at O, the origin of the Cartesian coordinate plane, and I walk to the point A, then my position has changed, and this is represented by the vector $\overrightarrow{OA}$.

It is possible to perform operations on vectors; this is called *vector algebra*. For one, we can *add* vectors by combining their changes in position. For example, if I move from O to A (represented by $\overrightarrow{OA}$) and *then* move from A to another point B (represented by $\overrightarrow{AB}$), then overall, I moved from O to B (represented by $\overrightarrow{OB}$); we call this new vector the *sum* of the first two vectors; that is,

$$\overrightarrow{OA} + \overrightarrow{AB} \;=\; \overrightarrow{OB}.$$

(continued)

(continued) In this problem, we are given the components of $\overrightarrow{OA}$ and $\overrightarrow{OB}$. We can substitute these into the previous equation to solve for $\overrightarrow{AB}$ (note that the components of the sum or difference of a pair of vectors are calculated "component-wise"):

$$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB} \implies [3,4]+\overrightarrow{AB}=[2,2]$$

$$\implies \overrightarrow{AB} = [2,2]-[3,4] = [2-3,2-4] = \boxed{[-1,-2].}$$

(5) In [Figure 1.24 in Poole's book], A, B, C, D, E, and F are the vertices of a regular hexagon [centered] at the origin. Express each of the following vectors in terms of $\mathbf{a}=\overrightarrow{OA}$ and $\mathbf{b}=\overrightarrow{OB}$.

(a) $\overrightarrow{AB}$

Solution:

If I walk from O to A and back to O, then there will be no difference between the coordinates of my starting position and the coordinates of my final position, so $\overrightarrow{AO}+\overrightarrow{OA}$ is the "zero vector," denoted by $\mathbf{0}$. Consequently, $\overrightarrow{AO}=-\overrightarrow{OA}=-\mathbf{a}$.

On to the given vector: to get from A to B, I can walk from A to O (represented by $\overrightarrow{AO}=-\mathbf{a}$) and then walk from O to B (represented by $\overrightarrow{OB}=\mathbf{b}$). Hence, $\overrightarrow{AB} = \boxed{-\mathbf{a}+\mathbf{b}.}$

(b) $\overrightarrow{AD}$

Solution:

If a vector tells me to move right two units, I can do that from *anywhere* in the Cartesian plane; in other words, vectors are *portable*. $\overrightarrow{OD}$ is parallel to, in the same direction as, and the same length as $\overrightarrow{AO}$, so it is the *same* vector!

Therefore, to get from A to D, I can walk from A to O (represented by $-\mathbf{a}$) and then walk from O to D (*also* $-\mathbf{a}$). Accordingly, $\overrightarrow{AB} = -\mathbf{a}+(-\mathbf{a}) = \boxed{-2\mathbf{a}.}$

(c) $\overrightarrow{BC}$

Solution:

$\overrightarrow{BC}$ is parallel to, in the same direction as, and the same length as $\overrightarrow{AO}$, so $\overrightarrow{BC} \;=\; \boxed{-\mathbf{a}.}$

(d) $\overrightarrow{CF}$

Solution:

Both $\overrightarrow{CO}$ and $\overrightarrow{OF}$ are the same as $\overrightarrow{BA}$, which is $-\overrightarrow{AB}$. Therefore,

$$\overrightarrow{CF} \;=\; \overrightarrow{CO}+\overrightarrow{OF} \;=\; -\overrightarrow{AB}+\left(-\overrightarrow{AB}\right) \;=\; -2\overrightarrow{AB} \;=\; -2(-\mathbf{a}+\mathbf{b}) \;=\; \boxed{2\mathbf{a}-2\mathbf{b}.}$$

(e) $\overrightarrow{AC}$

Solution:

$$\overrightarrow{AC} \;=\; \overrightarrow{AO}+\overrightarrow{OC} \;=\; \overrightarrow{AO}+\overrightarrow{OB}+\overrightarrow{BC} \;=\; \overrightarrow{AO}+\overrightarrow{OB}+\overrightarrow{AO} \;=\; 2\overrightarrow{AO}+\overrightarrow{OB}$$

$$=\; 2(-\mathbf{a})+\mathbf{b} \;=\; \boxed{-2\mathbf{a}+\mathbf{b}.}$$

(f) $\overrightarrow{BC}+\overrightarrow{DE}+\overrightarrow{FA}$

Solution:

$$\overrightarrow{BC}+\overrightarrow{DE}+\overrightarrow{FA} \;=\; \overrightarrow{BC}+\overrightarrow{CO}+\overrightarrow{OB} \;=\; \overrightarrow{BO}+\overrightarrow{OB} \;=\; \overrightarrow{BB} \;=\; \mathbf{0} \;=\; \boxed{0\mathbf{a}+0\mathbf{b}.}$$