MATH 1410
Solutions for Homework 3
(The mistake in the fifth line on page 10 has been corrected)
Submitted Friday, February 1, 2013

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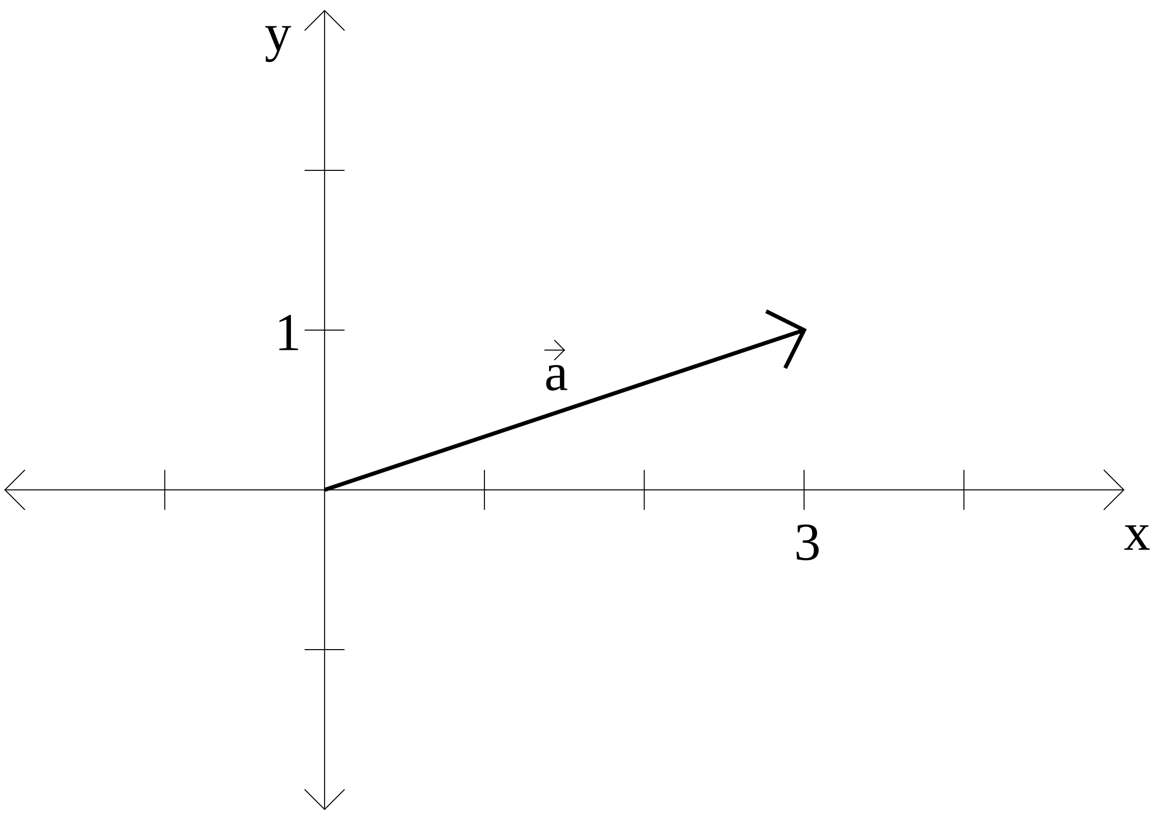
(1) Draw the following vectors in standard position (i.e. starting at the origin).

(a) $\vec{\mathbf{a}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

<u>Solution</u>:

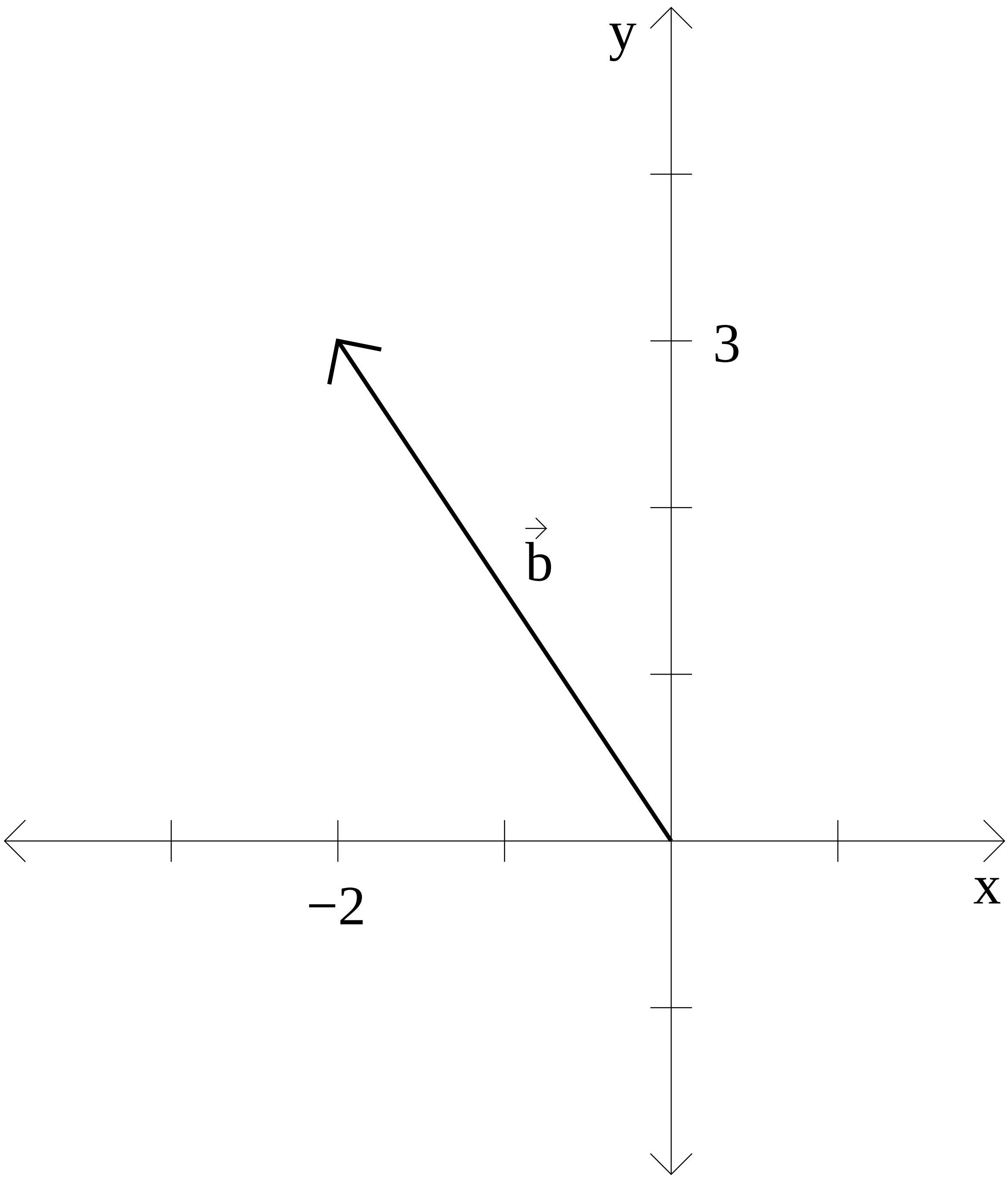
We read the vector from top to bottom: the first component, 3, is the change in the x-value (the horizontal position), and the second component, 1, is the change in the y-value (the vertical position). We know that the starting point (or "tail") of the vector is the origin (the point at which the x- and y-axes meet); to find its terminal point (or "tip"), we move 3 units right and 1 unit up. To draw the vector, we connect the tail and tip with a straight line segment and put an arrowhead at the tip, as follows:



(b) $\overrightarrow{\mathbf{b}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Solution:

If the change in a coordinate is negative, then we move in the opposite direction; in particular, if the first component is negative, then we move left, and if the second component is negative, then we move down. Consequently, to find the terminal point for the vector $\overrightarrow{\mathbf{b}}$ in standard position, we start at the origin and move 2 units left and 3 units up:



(c) $2\overrightarrow{\mathbf{a}} + 3\overrightarrow{\mathbf{b}}$

Solution:

There are two ways to draw this vector, both involving vector algebra:

Method 1: Geometrically:

Adding vectors combines their changes in position. In this case, we have

$$2\overrightarrow{\mathbf{a}} + 3\overrightarrow{\mathbf{b}} \;=\; \overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{b}}.$$

We start at the origin. The first $\overrightarrow{\mathbf{a}}$ takes us to $(3, 1)$. The second $\overrightarrow{\mathbf{a}}$ adds 3 more to our x-value and adds 1 more to our y-value, bringing us to $(6, 2)$. The first $\overrightarrow{\mathbf{b}}$ subtracts 2 from our x-value 6, making it 4, and adds 3 to our y-value 2, making it 5, meaning that we are now at $(4, 5)$. The second $\overrightarrow{\mathbf{b}}$ subtracts 2 more from our x-value and adds 3 more to our y-value, moving us to $(2, 8)$. The final vector, the third $\overrightarrow{\mathbf{b}}$, decreases our x-value from 2 to 0 and increases our y-value from 8 to 11, so the terminal point of $2\overrightarrow{\mathbf{a}} + 3\overrightarrow{\mathbf{b}}$ in standard position is $(0, 11)$. We then draw the vector as usual (the result may be found on the next page). Notice how the $\overrightarrow{\mathbf{a}}$'s and $\overrightarrow{\mathbf{b}}$'s were combined: the tip of one vector becomes the tail of the next. As a result, we sometimes call this the "tip-to-tail method."

What can we say about the vector $2\overrightarrow{\mathbf{a}} = \overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{a}}$? It is parallel with, in the same direction as, and *twice* the length of $\overrightarrow{\mathbf{a}}$. Similarly, $3\overrightarrow{\mathbf{b}} = \overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{b}}$ is parallel with, in the same direction as, and *three* times as long as $\overrightarrow{\mathbf{b}}$. This is the motivation for the next vector operation: "scalar multiplication," or the product of a vector with a number. Since $-\overrightarrow{\mathbf{a}}$ is in the *opposite* direction of $\overrightarrow{\mathbf{a}}$, we need to take care defining this operation:

Scalar Multiplication: Given any number c and any vector $\overrightarrow{\mathbf{v}}$, $c\overrightarrow{\mathbf{v}}$ is a vector that is parallel with $\overrightarrow{\mathbf{v}}$, $|c|$ times as long as $\overrightarrow{\mathbf{v}}$, and either in the same direction as $\overrightarrow{\mathbf{v}}$ if c is positive or in the opposite direction of $\overrightarrow{\mathbf{v}}$ if c is negative (if c is zero, then the scalar product is the zero vector, which has *no* direction).

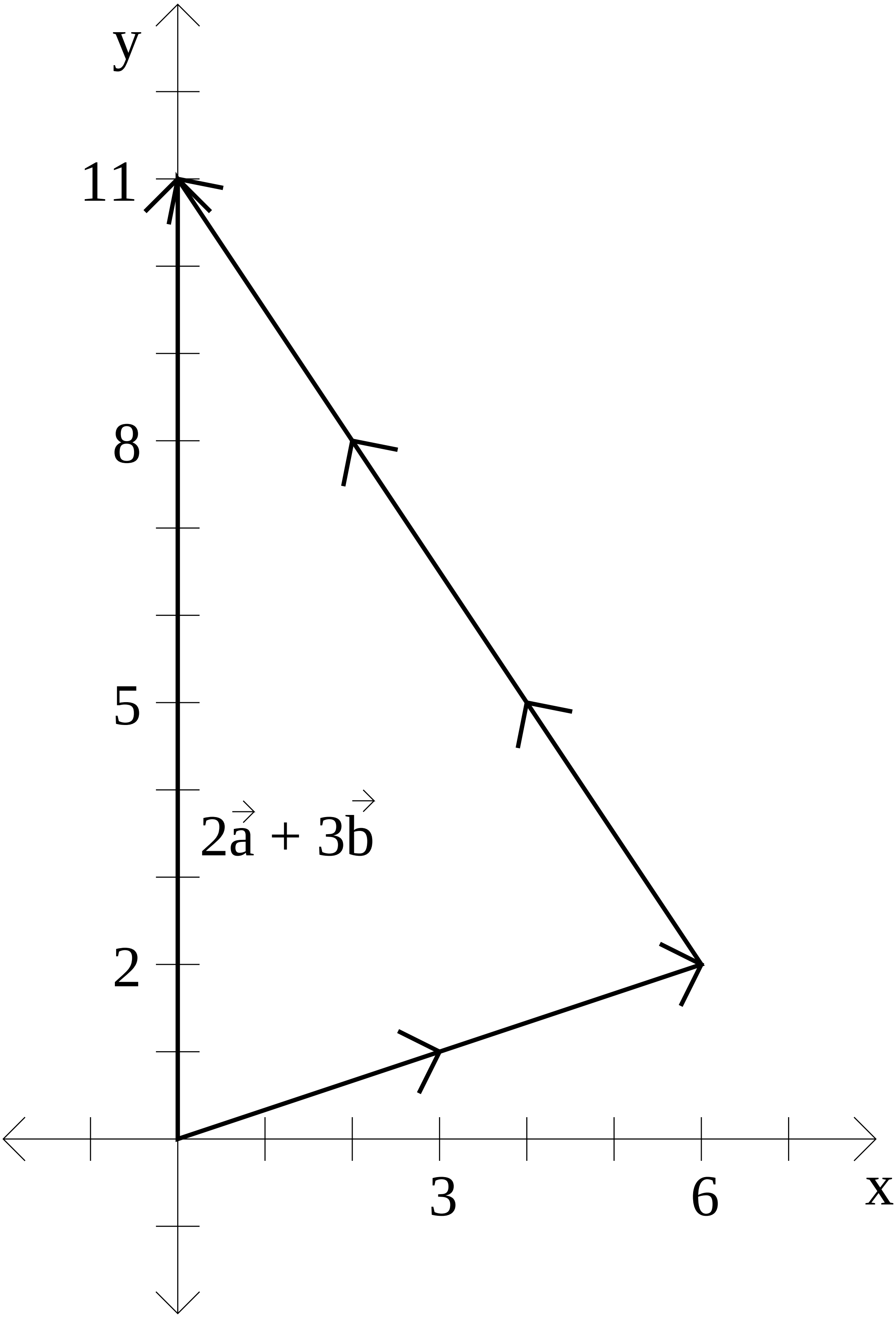
Method 2: Algebraically:

We arithmetically determine the components of $2\overrightarrow{\mathbf{a}} + 3\overrightarrow{\mathbf{b}}$, then draw the resulting vector in standard position. To scalar multiply c and $\overrightarrow{\mathbf{v}}$, we multiply each component of $\overrightarrow{\mathbf{v}}$ by c:

$$2\overrightarrow{\mathbf{a}} + 3\overrightarrow{\mathbf{b}} \;=\; 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 3 \end{bmatrix} \;=\; \begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} -6 \\ 9 \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 11 \end{bmatrix}.$$

(continued)

(continued) We may now draw this vector as usual:



(2) Let $\overrightarrow{\mathbf{u}} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and let $\overrightarrow{\mathbf{v}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

(a) Solve for $\overrightarrow{\mathbf{x}}$ where $2\overrightarrow{\mathbf{x}} + \overrightarrow{\mathbf{u}} = 3\overrightarrow{\mathbf{v}}$.

Solution:

Let's isolate $\overrightarrow{\mathbf{x}}$ first and *then* substitute $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$:

$$2\overrightarrow{\mathbf{x}} + \overrightarrow{\mathbf{u}} = 3\overrightarrow{\mathbf{v}} \implies 2\overrightarrow{\mathbf{x}} = 3\overrightarrow{\mathbf{v}} - \overrightarrow{\mathbf{u}} \implies \frac{1}{2}\left(2\overrightarrow{\mathbf{x}}\right) = \frac{1}{2}\left(3\overrightarrow{\mathbf{v}} - \overrightarrow{\mathbf{u}}\right)$$

$$\implies \overrightarrow{\mathbf{x}} = \frac{3}{2}\overrightarrow{\mathbf{v}} - \frac{1}{2}\overrightarrow{\mathbf{u}} = \frac{3}{2}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$\implies \overrightarrow{\mathbf{x}} = \begin{bmatrix} 0 \\ 3/2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/2 \\ 6/2 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 2/2 \\ 3/2 \end{bmatrix} = \boxed{\begin{bmatrix} 1/2 \\ 1/2 \\ 3/2 \end{bmatrix}.}$$

(b) Decide if $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ or not.

Solution:

A *linear combination* of a set of vectors (all from $\mathbb{R}^n$) is obtained by multiplying the vectors in that set by numbers and adding the results. For example,

$$2\overrightarrow{\mathbf{u}} + 3\overrightarrow{\mathbf{v}} = 2\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 12 \end{bmatrix}$$

is a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$. Note that we can call $[-2, 7, 12]$ a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ *without* having to identify the coefficients of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ (2 and 3, respectively) that we used to get it. Accordingly, a given vector *is* a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ *if and only if* it is the sum of scalar multiples of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$. What scalar multiples of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ would we add together to get the zero vector? Let's try...zero multiples!

$$0\overrightarrow{\mathbf{u}} + 0\overrightarrow{\mathbf{v}} = 0\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, the given vector $\boxed{\textit{is} \text{ a linear combination of } \overrightarrow{\mathbf{u}} \text{ and } \overrightarrow{\mathbf{v}}.}$

(c) Decide if $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ is a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ or not.

<u>**Solution:**</u>

This time, it is not obvious what scalar multiples of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ we would add together to obtain the given vector, so let us represent these coefficients by unknowns (say, c_1 and c_2). Then, the given vector is a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ if we can find values for c_1 and c_1 such that

$$c_1 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

To find these values, we first simplify the left-hand side of the equation using vector algebra:

$$\begin{bmatrix} -c_1 \\ 2c_1 \\ 3c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \implies \begin{bmatrix} -c_1 \\ 2c_1 + c_2 \\ 3c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

Two vectors are equal if and only if their corresponding components are equal; that is, their first components have to be the same (which in our case means that $-c_1$ must be equal to 3), their second components must be the same (meaning that $2c_1 + c_2$ must be equal to -1), their third components must be the same (so $3c_1 + 2c_2 = 1$), and so on. If we write down the equations that result, we obtain a [drumroll, please!] *system of linear equations!*

$$\begin{cases} ① : & -c_1 & & & = & 3 \\ ② : & 2c_1 & + & c_2 & = & -1 \\ ③ : & 3c_1 & + & 2c_2 & = & 1. \end{cases}$$

Clearly, we need to solve it. From ①, we see that $c_1 = -3$. Substituting this into ②, we get

$$2c_1 + c_2 = -1 \implies c_2 = -1 - 2c_1 = -1 - 2(-3) = -1 + 6 = 5.$$

We should check that these values satisfy ③. Even better, let us calculate the associated linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ (recall what c_1 and c_2 represented):

$$-3\overrightarrow{\mathbf{u}} + 5\overrightarrow{\mathbf{v}} = -3 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -9 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

This is the given vector, so it *is* a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$.

(d) Decide if $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ or not.

Solution:

As we did in part (b), we equate the given vector to the sum of scalar multiples of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ where the coefficients are unknowns (let's use c_1 and c_2 again):

$$c_1 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Like before, we use vector algebra to simplify the left-hand side:

$$\begin{bmatrix} -c_1 \\ 2c_1 \\ 3c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \implies \begin{bmatrix} -c_1 \\ 2c_1 + c_2 \\ 3c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Equating these vectors means equating their corresponding components, which results in this system of linear equations:

$$\begin{cases} \textcircled{1}: & -c_1 & & & = & 1 \\ \textcircled{2}: & 2c_1 & + & c_2 & = & 1 \\ \textcircled{3}: & 3c_1 & + & 2c_2 & = & 1. \end{cases}$$

Again, we solve it. From ①, we find that $c_1 = -1$. Substituting this into ②, we get

$$2c_1 + c_2 = 1 \implies c_2 = 1 - 2c_1 = 1 - 2(-1) = 1 + 2 = 3.$$

We now check that these values satisfy ③:

$$3c_1 + 2c_2 \ = \ 3(-1) + 2(3) \ = \ -3 + 6 \ = \ 3 \ \neq \ 1.$$

Ooops! They *don't* satisfy ③! That means that this system has *no* solution. (O_o)

Apparently, there are *no* scalar multiples of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ that we can add together to obtain the given vector.

We therefore conclude that the given vector is **[** *not* a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$. **]**

(3) Determine $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ and the projection of $\vec{\mathbf{u}}$ onto $\vec{\mathbf{v}}$, where

$$\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{\mathbf{v}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

<u>Solution</u>:

The *dot product* of vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, denoted by $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$, is the sum of the products of their corresponding components; that is, we multiply their first components, then multiply the second components, and so on, and add the results

$$\text{i.e. } \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = (1)(-1) + (2)(3) = -1+6 = \boxed{5.}$$

Next, the *projection of* $\vec{\mathbf{u}}$ *onto* $\vec{\mathbf{v}}$ is denoted and calculated by

$$\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) = \left(\frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}\right)\vec{\mathbf{v}}.$$

Notice that $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$ appears in the formula. Let's do this calculation separately:

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = (-1)(-1) + (3)(3) = (-1)^2 + 3^2 = 1+9 = 10.$$

Notice that we got this by squaring each component of $\vec{\mathbf{v}}$ and adding the results. In general, the dot product of a vector with itself is the sum of the squares of its components.

With these computations done, we may now determine the projection of $\vec{\mathbf{u}}$ onto $\vec{\mathbf{v}}$:

$$\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) = \left(\frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}\right)\vec{\mathbf{v}} = \left(\frac{5}{10}\right)\vec{\mathbf{v}} = \frac{1}{2}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}.}$$

(4) Determine $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ and the projection of $\vec{\mathbf{u}}$ onto $\vec{\mathbf{v}}$, where

$$\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{\mathbf{v}} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}.$$

<u>Solution</u>:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = (1)(-1) + (2)(3) + (3)(-1) = -1+6-3 = \boxed{2.}$$

(continued)

(continued) Next, $\overrightarrow{\mathbf{v}} \cdot \overrightarrow{\mathbf{v}} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = (-1)^2 + 3^2 + (-1)^2 = 1+9+1 = 11.$

Hence, $\text{proj}_{\overrightarrow{\mathbf{v}}}(\overrightarrow{\mathbf{u}}) = \left(\dfrac{\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}}}{\overrightarrow{\mathbf{v}} \cdot \overrightarrow{\mathbf{v}}}\right)\overrightarrow{\mathbf{v}} = \left(\dfrac{2}{11}\right)\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} -2/11 \\ 6/11 \\ -2/11 \end{bmatrix}}.$

(5) Show that for any pair of vectors $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ in $\mathbb{R}^n$ we have

$$(\overrightarrow{\mathbf{u}} - \overrightarrow{\mathbf{v}}) \cdot (\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}}) = \|\overrightarrow{\mathbf{u}}\|^2 - \|\overrightarrow{\mathbf{v}}\|^2.$$

Solution:

The *length* of a vector $\overrightarrow{\mathbf{u}} = [u_1, u_2, \ldots u_n]$ is denoted and calculated by

$$\|\overrightarrow{\mathbf{u}}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

In other words, it is the square root of the sum of the squares of its components. We discovered earlier that the dot product of a vector with itself was the sum of the squares of its components. Consequently,

$$\|\overrightarrow{\mathbf{u}}\| = \sqrt{\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{u}}} \implies \|\overrightarrow{\mathbf{u}}\|^2 = \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{u}},$$

which means that the dot product of a vector with itself is the square of its length.

To prove the given equation, we will work on the left-hand side until it looks like the right-hand side. What can we possibly do with the left-hand side? Well, some properties of vector algebra are incredibly similar to those of real numbers. For example, in vector algebra, the dot product is distributive over vector addition, which is a complicated way of saying that for all vectors $\overrightarrow{\mathbf{u}}$, $\overrightarrow{\mathbf{v}}$, and $\overrightarrow{\mathbf{w}}$ in $\mathbb{R}^n$,

$$\overrightarrow{\mathbf{u}} \cdot (\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}}) = \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}},$$

which is just like the distributivity of multiplication over addition in the real numbers. For a full list of these properties, please refer to Theorems 1.1 and 1.2 in Poole's book. On to the proof: let $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ be any vectors in $\mathbb{R}^n$. Then,

$$(\overrightarrow{\mathbf{u}} - \overrightarrow{\mathbf{v}}) \cdot (\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}}) = (\overrightarrow{\mathbf{u}} + (-\overrightarrow{\mathbf{v}})) \cdot (\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}}) \qquad \text{(definition of vector subtraction)}$$

$$= (\overrightarrow{\mathbf{u}} + (-\overrightarrow{\mathbf{v}})) \cdot \overrightarrow{\mathbf{u}} + (\overrightarrow{\mathbf{u}} + (-\overrightarrow{\mathbf{v}})) \cdot \overrightarrow{\mathbf{v}} \qquad \text{(Theorem 1.2b)}$$

(continued)

$$\text{(continued)} = \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} + \left(-\vec{\mathbf{v}}\right) \cdot \vec{\mathbf{u}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \left(-\vec{\mathbf{v}}\right) \cdot \vec{\mathbf{v}} \qquad \text{(Theorems 1.2a and 1.2b)}$$

$$= \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} + \left(-\vec{\mathbf{v}} \cdot \vec{\mathbf{u}}\right) + \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \left(-\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}\right) \qquad \text{(Theorem 1.2c)}$$

$$= \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} - \vec{\mathbf{v}} \cdot \vec{\mathbf{u}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \qquad \text{(definition of vector subtraction)}$$

$$= \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} - \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \qquad \text{(Theorem 1.2a)}$$

$$= \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} - \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \left\|\vec{\mathbf{u}}\right\|^2 - \left\|\vec{\mathbf{v}}\right\|^2 \qquad \text{(cancellation and the result that we discovered earlier).}$$

We obtained the right-hand side of the given equation, which proves that it is true.

(6) Show that there are no vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ such that $\left\|\vec{\mathbf{u}}\right\| = 1$, $\left\|\vec{\mathbf{v}}\right\| = 2$, and $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 3$.

Solution:

There are *two* ways to do this:

Method 1: The Cauchy-Schwarz Inequality:

According to this theorem (1.4 in Poole's book), for all vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in $\mathbb{R}^n$,

$$\left|\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}\right| \leq \left\|\vec{\mathbf{u}}\right\| \left\|\vec{\mathbf{v}}\right\|.$$

For the given vectors, however,

$$\left|\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}\right| = |3| = 3 \not\leq 2 = (1)(2) = \left\|\vec{\mathbf{u}}\right\| \left\|\vec{\mathbf{v}}\right\|.$$

The theorem isn't wrong (it is a *theorem*, afterall), so there *cannot* be any such vectors.

Method 2: The Angle Formula:

For any non-zero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in $\mathbb{R}^n$, the angle θ[†] that they form satisfies the equation

$$\cos\theta = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\left\|\vec{\mathbf{u}}\right\| \left\|\vec{\mathbf{v}}\right\|}.$$

For the given vectors, $\cos\theta = 3/((1)(2)) = 1.5$, which is *impossible:* the cosine of an angle cannot exceed 1! Ergo, there *cannot* be any such vectors.

[†] This is the Greek letter "theta."