MATH 1410
Solutions for Homework 4
Submitted Friday, February 15, 2013

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(1) Row reduce the following matrices. (Make sure you specify your row operations.)

(a) $\begin{bmatrix} 1 & 5 & 6 & 2 \\ 2 & 11 & 11 & 5 \\ 3 & 10 & 20 & 5 \end{bmatrix}$

Solution:

Let R_i denote row number i. There are three row operations that we may use to row reduce a matrix, called the *elementary row operations:*

1. $R_i \leftrightarrow R_j$, which interchanges rows i and j.

2. $k\,R_i$, which multiplies row i by a nonzero number k.

3. $R_i \pm k\,R_j$, which adds (or subtracts) k times row j to (or from) row i (note that row j is *not* changed by this operation).

Our goal in using these operations is to make the matrix satisfy these two conditions:

1. Any rows of zeros are at the bottom.

2. The first nonzero number in each row (called its *leading entry)* is to the left of any leading entries below.

A matrix that satisfies both of these conditions is said to be in *row echelon form.* In the given matrix, the leading entry in first row is *not* to the left of the leading entries in the second and third rows, so we will start this process by turning the second and third entries in the first column into zeros:

$$\begin{bmatrix} 1 & 5 & 6 & 2 \\ 2 & 11 & 11 & 5 \\ 3 & 10 & 20 & 5 \end{bmatrix}$$

$$\overset{R_2 - 2R_1}{\longrightarrow} \begin{bmatrix} 1 & 5 & 6 & 2 \\ 0 & 1 & -1 & 1 \\ 3 & 10 & 20 & 5 \end{bmatrix}$$

(continued)

(continued)

$$= \quad \begin{bmatrix} 1 & 5 & 6 & 2 \\ 0 & 1 & -1 & 1 \\ 3 & 10 & 20 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 - 3R_1} \quad \begin{bmatrix} 1 & 5 & 6 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & -5 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 + 5R_2} \quad \boxed{\begin{bmatrix} 1 & 5 & 6 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & 4 \end{bmatrix}.}$$

The matrix is now in row echelon form. Note that the row echelon form of a matrix is *not* unique! For example,

$$\boxed{\begin{bmatrix} 1 & -1 & 9 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix}}$$

is *another* row echelon form of the given matrix, obtained using this sequence of elementary row operations: $R_1 \leftrightarrow R_3$, $R_1 - R_2$, $R_2 - 2R_1$, $R_3 - R_1$, $R_2 - 2R_3$, and $R_3 - 6R_2$.

(b) $\begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 7 \\ 5 & 10 & 2 \end{bmatrix}$

<u>Solution</u>:

It's easier to make an entry zero by adding or subtracting a multiple of *one* than, say, two, so this time we'll begin by multiplying the first row by $1/2^{\dagger}$, thereby making its leading entry one:

$$\begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 7 \\ 5 & 10 & 2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1} \quad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 7 \\ 5 & 10 & 2 \end{bmatrix}$$

(continued)

†To *divide* a row by a nonzero number (say, k), we *multiply* that row by the *reciprocal* of the number i.e. $1/k$.

(continued)

$$= \quad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 7 \\ 5 & 10 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 - 3R_1} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 5 & 10 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 - 5R_1} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & 0 & -13 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -13 \end{bmatrix}$$

$$\xrightarrow{R_3 + 13\,R_2} \quad \boxed{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.}$$

(c) $\begin{bmatrix} 7 & 3 \\ 5 & 9 \\ 1 & -2 \\ 3 & 0 \end{bmatrix}$

Solution:

We can make the leading entry in the first row a one by *interchanging* rows:

$$\begin{bmatrix} 7 & 3 \\ 5 & 9 \\ 1 & -2 \\ 3 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \quad \begin{bmatrix} 1 & -2 \\ 5 & 9 \\ 7 & 3 \\ 3 & 0 \end{bmatrix}$$

(continued)

(continued)

$$= \begin{bmatrix} 1 & -2 \\ 5 & 9 \\ 7 & 3 \\ 3 & 0 \end{bmatrix}$$

$$\underset{\longrightarrow}{R_2 - 5R_1} \begin{bmatrix} 1 & -2 \\ 0 & 19 \\ 7 & 3 \\ 3 & 0 \end{bmatrix}$$

$$\underset{\longrightarrow}{R_3 - 7R_1} \begin{bmatrix} 1 & -2 \\ 0 & 19 \\ 0 & 17 \\ 3 & 0 \end{bmatrix}$$

$$\underset{\longrightarrow}{R_4 - 3R_1} \begin{bmatrix} 1 & -2 \\ 0 & 19 \\ 0 & 17 \\ 0 & 6 \end{bmatrix}$$

$$\underset{\longrightarrow}{\frac{1}{19} R_2} \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 17 \\ 0 & 6 \end{bmatrix}$$

$$\underset{\longrightarrow}{R_3 - 17R_2} \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\underset{\longrightarrow}{R_4 - 6R_2} \boxed{\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.}$$

(2) Solve the following system of equation[s]:

$$\begin{array}{rcrcrcr} 2a & + & 3b & + & 4c & = & 5 \\ a & + & 2b & + & c & = & 0 \\ 3a & + & b & + & 2c & = & -4 \end{array}$$

<u>**Solution**</u>**:**

We begin by creating the associated *augmented matrix,* in which each row represents an equation in the system, each column but the last contains the coefficients of a specific variable, and the last column contains the constants:

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 5 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & 2 & -4 \end{array}\right]$$

Now for the *pièce de résistance:* performing an elementary row operation on an augmented matrix may change the underlying system, but it *preserves* the (possibly nonexistent) solution set! So, if we row reduce an augmented matrix and solve the resulting underlying system, we will have solved the original system!

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 5 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & 2 & -4 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & -4 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 5 \\ 3 & 1 & 2 & -4 \end{array}\right]$$

$$\xrightarrow{R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 5 \\ 0 & -5 & -1 & -4 \end{array}\right]$$

$$\xrightarrow{R_3 - 5R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 5 \\ 0 & 0 & -11 & -29 \end{array}\right].$$

(continued)

(continued) The matrix is in row echelon form, so we may now write down the underlying system:

$$\begin{array}{rcrcrcr} a & + & 2b & + & c & = & 0 \\ & & -b & + & 2c & = & 5 \\ & & & & -11c & = & -29 \end{array}$$

To solve this system, we do *back-substitution:* we solve the last equation for the last variable, then substitute that value into the second-last equation to solve for the second-last variable, then substitute both known values into the third-last equation to solve for the third-last variable, and so on.

To begin, $-11c = -29 \implies c = \frac{-29}{-11} = \frac{29}{11}$.

Next, $-b+2c = 5 \implies -b = -2c+5 \implies b = 2c-5 = 2\left(\frac{29}{11}\right) - \frac{5}{1} = \frac{58}{11} - \frac{55}{11} = \frac{3}{11}$.

Lastly, $a+2b+c = 0 \implies a = -2b-c = -2\left(\frac{3}{11}\right) - \left(\frac{29}{11}\right) = -\frac{6}{11} - \frac{29}{11} = -\frac{35}{11}$.

Consequently, the solution for both this and (more importantly) the original system is

$$(a, b, c) = \boxed{\left(-\frac{35}{11}, \frac{3}{11}, \frac{29}{11}\right).}$$