MATH 1410
Solutions for Homework 5
Submitted Friday, March 1, 2013

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(1) Row reduce the following matrices to [reduced row] echelon form. (Make sure you specify your row operations.)

(a) $\begin{bmatrix} 3 & 5 & 6 & 1 & 0 \\ 2 & 11 & 11 & 5 & 0 \\ 3 & 10 & 20 & 5 & 0 \end{bmatrix}$

Solution:

A matrix is in *reduced row echelon form* if it satisfies these three conditions:

1. It is in row echelon form.

2. The leading entry in each nonzero row is a 1 (called a *leading 1*).

3. In every column that contains a leading 1, the other entries are all zeros.

Of course, we will use the elementary row operations to obtain this form. To begin, we want the leading entry in the first row to be a 1 (right now it is a 3). We could multiply row 1 by 1/3, but that would introduce fractions.† A nicer alternative is to subtract row 2 from row 1 (since $3-2=1$):

$$\begin{bmatrix} 3 & 5 & 6 & 1 & 0 \\ 2 & 11 & 11 & 5 & 0 \\ 3 & 10 & 20 & 5 & 0 \end{bmatrix}$$

$$\xrightarrow{\mathrm{R}_1 - \mathrm{R}_2} \begin{bmatrix} 1 & -6 & -5 & -4 & 0 \\ 2 & 11 & 11 & 5 & 0 \\ 3 & 10 & 20 & 5 & 0 \end{bmatrix}.$$

Hmmm...this might take a while. Can we do more than one operation at a time? Yes, *provided* that (a) we don't change a row more than once and (b) we don't change a row that changes another row. For example, we can't do $\mathrm{R}_3 - \mathrm{R}_1$ and $\mathrm{R}_3 - \mathrm{R}_2$ in a single step because we'd be changing row 3 more than once. Also, we can't do both $\mathrm{R}_2 + \mathrm{R}_1$ and $\mathrm{R}_3 - \mathrm{R}_2$ because we'd be changing a row (R_2) that we would then use to change another row (R_3); a natural question for a grader to ask in this scenario is, "*Which* row 2 did you subtract from row 3?"

(continued)

†It might be inevitable that fractions will appear, but we can at least postpone their appearance.

(continued) What about $R_2 - 2R_1$ and $R_3 - 3R_1$? Row 2 gets changed *once,* row 3 gets changed *once,* and row 1, the row being used to change the other rows, does *not* be changed; in other words, the rules are not broken, so we *can* do both of these operations in one step:

$$\begin{bmatrix} 1 & -6 & -5 & -4 & 0 \\ 2 & 11 & 11 & 5 & 0 \\ 3 & 10 & 20 & 5 & 0 \end{bmatrix}$$

$$\begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & -6 & -5 & -4 & 0 \\ 0 & 23 & 21 & 13 & 0 \\ 0 & 28 & 35 & 17 & 0 \end{bmatrix} \ldots$$

You know, there may be some inventive way of turning the 23 into a 1 without dividing, but I suspect that it requires a LOT of steps, so let's just divide and deal with the resulting fractions:

$$\begin{bmatrix} 1 & -6 & -5 & -4 & 0 \\ 0 & 23 & 21 & 13 & 0 \\ 0 & 28 & 35 & 17 & 0 \end{bmatrix}$$

$$\begin{matrix} \frac{1}{23} R_2 \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & -6 & -115/23 & -92/23 & 0 \\ 0 & 1 & 21/23 & 13/23 & 0 \\ 0 & 28 & 805/23 & 391/23 & 0 \end{bmatrix}^{*}$$

$$\begin{matrix} R_1 + 6R_2 \\ R_3 - 28R_2 \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 11/23 & -14/23 & 0 \\ 0 & 1 & 21/23 & 13/23 & 0 \\ 0 & 0 & 217/23 & 27/23 & 0 \end{bmatrix}$$

$$\begin{matrix} \frac{1}{217} R_3 \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 11/23 & -3038/4991 & 0 \\ 0 & 1 & 21/23 & 2821/4991 & 0 \\ 0 & 0 & 1/23 & 27/4991 & 0 \end{bmatrix}^{\ddagger}$$

$$\begin{matrix} R_1 - 11R_3 \\ R_2 - 21R_3 \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -3335/4991 & 0 \\ 0 & 1 & 0 & 2254/4991 & 0 \\ 0 & 0 & 1/23 & 27/4991 & 0 \end{bmatrix}$$

$$\begin{matrix} 23R_3 \\ \longrightarrow \end{matrix} \boxed{\begin{bmatrix} 1 & 0 & 0 & -3335/4991 & 0 \\ 0 & 1 & 0 & 2254/4991 & 0 \\ 0 & 0 & 1 & 27/217 & 0 \end{bmatrix}.} \qquad \S$$

*Notice that I converted some of the entries in the *other* rows into fractions whose denominators are 23 by multiplying each one by 23/23; this will make adding and subtracting multiples of these numbers easier.

‡Here, I multiplied some entries by 217/217, again to simplify their addition and subtraction.

§This is from Soroosh: the fraction 2254/4991 and −3335/4991 can be simplified, although it is not obvious at all that this is the case, and in fact the only reason I know that is that my computer told this to me.

(b) $\left[\begin{array}{rrrrrr} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 7 & 3 & 0 & 1 & 0 \\ -3 & 10 & -5 & 0 & 0 & 1 \end{array}\right]$

Solution:

The leading entry in the first row is already a 1 (yay!) so we may proceed to eliminate the other nonzero entries in the first column:

$$\left[\begin{array}{rrrrrr} \textcircled{1} & 4 & 2 & 1 & 0 & 0 \\ 2 & 7 & 3 & 0 & 1 & 0 \\ -3 & 10 & -5 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_2 - 2R_1 \\ R_3 + 3R_1 \\ \longrightarrow \end{array} \left[\begin{array}{rrrrrr} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 22 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} -R_2 \\ \longrightarrow \end{array} \left[\begin{array}{rrrrrr} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 1 & 2 & -1 & 0 \\ 0 & 22 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 - 4R_2 \\ R_3 - 22R_2 \\ \longrightarrow \end{array} \left[\begin{array}{rrrrrr} 1 & 0 & -2 & -7 & 4 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -21 & -41 & 22 & 1 \end{array}\right]$$

$$\begin{array}{c} -\frac{1}{21}R_3 \\ \longrightarrow \end{array} \left[\begin{array}{rrrrrr} 1 & 0 & -2 & -147/21 & 84/21 & 0 \\ 0 & 1 & 1 & 42/21 & -21/21 & 0 \\ 0 & 0 & \textcircled{1} & 41/21 & -22/21 & -1/21 \end{array}\right]$$

$$\begin{array}{c} R_1 + 2R_3 \\ R_2 - R_3 \\ \longrightarrow \end{array} \boxed{\left[\begin{array}{rrrrrr} 1 & 0 & 0 & -65/21 & 40/21 & -2/21 \\ 0 & 1 & 0 & 1/21 & 1/21 & 1/21 \\ 0 & 0 & 1 & 41/21 & -22/21 & -1/21 \end{array}\right].}$$

(2) Find the point (a, b, c, d) in $\mathbb{R}^4$ such that a, b, c, and d satisfy

$$\begin{array}{ccccccccc} 2a & + & 3b & + & 4c & + & 4d & = & 2 \\ 3a & + & 2b & + & 4c & + & 2d & = & 3 \\ 4a & + & 3b & + & 2c & + & 3d & = & 4 \end{array}$$

and it is closest to the origin. (Hint: The solution to the above system is a line, so the problem can be reduced to projecting a point onto a line.)

Solution:

Saying that the coordinates of the point satisfy the given equations is equivalent to saying that this point belongs to the *solution set* of the system formed by these equations. We therefore need to solve this system. As per usual, we create and row reduce the corresponding augmented matrix:

$$\left[\begin{array}{cccc|c} 2 & 3 & 4 & 4 & 2 \\ 3 & 2 & 4 & 2 & 3 \\ 4 & 3 & 2 & 3 & 4 \end{array}\right]$$

$$\overset{R_1 - R_2}{\longrightarrow} \left[\begin{array}{cccc|c} -1 & 1 & 0 & 2 & -1 \\ 3 & 2 & 4 & 2 & 3 \\ 4 & 3 & 2 & 3 & 4 \end{array}\right]$$

$$\overset{-R_1}{\longrightarrow} \left[\begin{array}{cccc|c} \textcircled{1} & -1 & 0 & -2 & 1 \\ 3 & 2 & 4 & 2 & 3 \\ 4 & 3 & 2 & 3 & 4 \end{array}\right]$$

$$\overset{\substack{R_2 - 3R_1 \\ R_3 - 4R_1}}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 1 \\ 0 & 5 & 4 & 8 & 0 \\ 0 & 7 & 2 & 11 & 0 \end{array}\right]$$

$$\overset{\frac{1}{5}R_2}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -10/5 & 1 \\ 0 & \textcircled{1} & 4/5 & 8/5 & 0 \\ 0 & 7 & 10/5 & 55/5 & 0 \end{array}\right]$$

$$\overset{\substack{R_1 + R_2 \\ R_3 - 7R_2}}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 4/5 & -2/5 & 1 \\ 0 & 1 & 4/5 & 8/5 & 0 \\ 0 & 0 & -18/5 & -1/5 & 0 \end{array}\right]$$

$$\overset{-\frac{1}{18}R_3}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 4/5 & -36/90 & 1 \\ 0 & 1 & 4/5 & 144/90 & 0 \\ 0 & 0 & 1/5 & 1/90 & 0 \end{array}\right]$$

(continued)

(continued)

$$= \left[\begin{array}{cccc|c} 1 & 0 & 4/5 & -36/90 & 1 \\ 0 & 1 & 4/5 & 144/90 & 0 \\ 0 & 0 & 1/5 & 1/90 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - 4R_3 \\ R_2 - 4R_3 \\ \longrightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -40/90 & 1 \\ 0 & 1 & 0 & 140/90 & 0 \\ 0 & 0 & 1/5 & 1/90 & 0 \end{array}\right]$$

$$\begin{array}{c} 5R_3 \\ \longrightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -4/9 & 1 \\ 0 & 1 & 0 & 14/9 & 0 \\ 0 & 0 & 1 & 1/18 & 0 \end{array}\right].$$

This is in reduced row echelon form, so we may now write down and solve the associated system:

$$\begin{array}{ccccccc} a & & & - & \frac{4}{9}d & = & 1 \\ & b & & + & \frac{14}{9}d & = & 0 \\ & & c & + & \frac{1}{18}d & = & 0 \end{array}$$

d is not a leading variable, and is therefore free to take any value i.e. $d = t$. Substituting this into the equations above and solving for each leading variable yields this solution:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 + (4/9)t \\ (-14/9)t \\ (-1/18)t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4/9 \\ -14/9 \\ -1/18 \\ 1 \end{bmatrix}.$$

As expected, we've obtained a line; specifically, one containing the point $P = (1, 0, 0, 0)$ and having direction vector $\overrightarrow{\mathbf{d}} = [4/9, -14/9, -1/18, 1]$. To project the origin, or $O = (0, 0, 0, 0)$, onto this line, we first need the vector

$$\overrightarrow{PO} = -\overrightarrow{OP} = -\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(continued)

(continued) Next, we compute $\vec{\mathbf{a}} = \operatorname{proj}_{\vec{\mathbf{d}}}\left(\overrightarrow{PO}\right) = \left(\dfrac{\overrightarrow{PO}\cdot\vec{\mathbf{d}}}{\vec{\mathbf{d}}\cdot\vec{\mathbf{d}}}\right)\vec{\mathbf{d}}$

$$= \left(\frac{\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}\cdot\begin{bmatrix}4/9\\-14/9\\-1/18\\1\end{bmatrix}}{\begin{bmatrix}4/9\\-14/9\\-1/18\\1\end{bmatrix}\cdot\begin{bmatrix}4/9\\-14/9\\-1/18\\1\end{bmatrix}}\right)\vec{\mathbf{d}} = \left(\frac{(-1)(4/9)+(0)(-14/9)+(0)(-1/18)+(0)(1)}{(4/9)^2+(-14/9)^2+(-1/18)^2+(1)^2}\right)\vec{\mathbf{d}}$$

$$= \left(\frac{(-4/9)+0+0+0}{(8/18)^2+(-28/18)^2+(-1/18)^2+(18/18)^2}\right)\vec{\mathbf{d}}$$

$$= \left(\frac{-4/9}{(64/324)+(784/324)+(1/324)+(324/324)}\right)\vec{\mathbf{d}} = \left(\frac{-4/9}{1173/324}\right)\vec{\mathbf{d}}$$

$$= \left(-\frac{4}{9}\cdot\frac{324}{1173}\right)\vec{\mathbf{d}} = -\frac{48}{391}\vec{\mathbf{d}} = -\frac{48}{391}\begin{bmatrix}4/9\\-14/9\\-1/18\\1\end{bmatrix} = \begin{bmatrix}-64/1173\\224/1173\\8/1173\\-48/391\end{bmatrix}.$$

Since $\vec{\mathbf{a}}$ is a vector, it represents a change in position; specifically, it represents the change in position from the point P to the point on the line that is closest to the origin. Therefore, the desired point (in vector form) is

$$\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \overrightarrow{OP}+\vec{\mathbf{a}} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}+\begin{bmatrix}-64/1173\\224/1173\\8/1173\\-48/391\end{bmatrix} = \boxed{\begin{bmatrix}1109/1173\\224/1173\\8/1173\\-48/391\end{bmatrix}.}$$

(3) Find x so that the vector $[\, x,\ 9,\ -14,\ 4\,]$ is a linear combination of the vectors

$$\begin{bmatrix} -2 \\ 1 \\ -1/2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

Solution:

The first given vector is a linear combination of the other given vectors if and only if we can find numbers c_1, c_2, and c_3 such that

$$c_1 \begin{bmatrix} -2 \\ 1 \\ -1/2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ 9 \\ -14 \\ 4 \end{bmatrix}$$

$$\implies \begin{bmatrix} -2c_1 \\ c_1 \\ -c_1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} c_2 \\ -c_2 \\ 3c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} c_3 \\ 0 \\ -c_3 \\ 2c_3 \end{bmatrix} = \begin{bmatrix} x \\ 9 \\ -14 \\ 4 \end{bmatrix}$$

$$\implies \begin{bmatrix} -2c_1 + c_2 + c_3 \\ c_1 - c_2 \\ (-c_1/2) + 3c_2 - c_3 \\ 2c_3 \end{bmatrix} = \begin{bmatrix} x \\ 9 \\ -14 \\ 4 \end{bmatrix} \implies \left\{ \begin{array}{rcrcrcr} -2c_1 & + & c_2 & + & c_3 & = & x \\ c_1 & - & c_2 & & & = & 9 \\ (-c_1/2) & + & 3c_2 & - & c_3 & = & -14 \\ & & & & 2c_3 & = & 4. \end{array} \right.$$

There are (at least) two ways to proceed from here:

Method 1: Solve for the c_i's first, then find x.

We see from the fourth equation that $c_3 = 2$, while the second equation tells us that $c_1 = 9 + c_2$. Substituting these into the third equation, we get

$$\frac{-c_1}{2} + 3c_2 - c_3 = -14 \implies -c_1 + 6c_2 - 2c_3 = -28 \implies -(9 + c_2) + 6c_2 - 2(2) = -28$$

$$\implies -9 - c_2 + 6c_2 - 4 = -28 \implies 5c_2 = -28 + 9 + 4 = -15 \implies c_2 = -\frac{15}{5} = -3.$$

Then, $c_1 = 9 + c_2 = 9 + (-3) = 6$, so according to the first equation,

$$x = -2c_1 + c_2 + c_3 = -2(6) + (-3) + (2) = -12 - 1 = \boxed{-13.}$$

Method 2: Create and row reduce the associated augmented matrix, treating x as a constant:

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & x \\ 1 & -1 & 0 & 9 \\ -1/2 & 3 & -1 & -14 \\ 0 & 0 & 2 & 4 \end{array}\right] \cdots$$

Tweety: [*looks at the columns in the augmented matrix*] I tawt I taw the vectors! [*looks at the columns again*] I did! I did taw the vectors!

Sean: That's right, Tweety! The *columns* in our augmented matrix are actually the *vectors* that were given to us; in particular, the solution column (the last column) is the vector that we want to express as a linear combination of the others, and those other vectors are the coefficient columns. Had we known this at the beginning, we could have created the augmented matrix right away! Anyways, let us now reduce this matrix:

$$\begin{array}{c} R_1 \leftrightarrow R_2 \\ 2R_3 \\ (1/2)R_4 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} \textcircled{1} & -1 & 0 & 9 \\ -2 & 1 & 1 & x \\ -1 & 6 & -2 & -28 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{array}{c} R_2 + 2R_1 \\ R_3 + R_1 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 9 \\ 0 & -1 & 1 & x+18 \\ 0 & 5 & -2 & -19 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ R_3 + 5R_2 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -x-9 \\ 0 & -1 & 1 & x+18 \\ 0 & 0 & 3 & 5x+71 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{array}{c} -R_2 \\ R_3 \leftrightarrow R_4 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -x-9 \\ 0 & \textcircled{1} & -1 & -x-18 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 3 & 5x+71 \end{array}\right]$$

$$\begin{array}{c} R_1 + R_3 \\ R_2 + R_3 \\ R_4 - 3R_3 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -x-7 \\ 0 & 1 & 0 & -x-16 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5x+65 \end{array}\right].$$

According to the equation represented by the last row in this matrix,

$$0 = 5x + 65 \implies -5x = 65 \implies x = -\frac{65}{5} = \boxed{-13.}$$