MATH 1410
Solutions for Homework 6
Submitted Friday, March 8, 2013

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(1) (a) Show that $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

Solution:

The *span* of vectors $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots \overrightarrow{\mathbf{v}}_k$ (all from $\mathbb{R}^n$) is the set of all their linear combinations. To say that their span *is* $\mathbb{R}^n$ is to say that *every* vector in $\mathbb{R}^n$ is a linear combination of these vectors.

For this problem, the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is $\mathbb{R}^2$ if and only if every vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

To determine if this is the case, we row reduce the augmented matrix whose columns are the vectors and whose last column is the vector that we want to check is a linear combination of the others:

$$\left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & -1 & y \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & -2 & y-x \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & 1 & (x/2)-(y/2) \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & (x/2)+(y/2) \\ 0 & 1 & (x/2)-(y/2) \end{array}\right].$$

This matrix is in reduced row echelon form. Because there is a leading one to the left of the partition in each of its rows, the associated system has a solution no matter what x and y are. Therefore, the span of the given vectors is indeed $\mathbb{R}^2$.

Note: We would have reached this conclusion *regardless* of what had happened in the last column; in other words, the last column is *expendable*. As a result, an alternative way to do such a problem is to row reduce the matrix whose columns are *just* the given vectors (we *omit* the last column containing an arbitrary vector from $\mathbb{R}^n$) and show that its reduced row echelon form has a leading one in every row.

(b) Show that $\mathbb{R}^3 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$.

<u>Solution</u>:

Let's use the alternative method suggested on the previous page i.e. we create and row reduce the matrix whose columns are just the vectors whose span is being taken:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 + R_2]{R_1 - R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 - R_3]{R_1 + R_3} \begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}.$$

The resulting reduced row echelon form matrix has a leading one in every row, so the given span is indeed $\mathbb{R}^3$.

(2) For any of the sets below, decide if they are linearly independent, or linearly dependent, and state why. If they are linearly dependent, find a dependence relationship among the vectors.

(a) $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

<u>Solution</u>:

Essentially, a set of vectors from $\mathbb{R}^n$ is called "linearly dependent" if and only if one of those vectors is a linear combination of the others; it is called "linearly independent" if and only if it is *not* linearly dependent.

In the case that the set contains only two vectors, saying that one of them is a linear combination of the other is equivalent to saying that it is a *scalar multiple* of the other.

Say that one of the given vectors is a scalar multiple of the other. Then, because their second components (their "y-values") are the same, the scalar must be 1, which implies that the vectors are equal. However, their first components (their "x-values") are different, so they are *not* equal. Consequently, neither vector is a scalar multiple of the other, which means that the given set is *not* linearly dependent.

Ergo, the given set of vectors is $\boxed{\text{linearly independent.}}$

(b) $\begin{bmatrix} 1 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}$

<u>Solution</u>:

Here is the *actual* definition[†]: a set of vectors $\overrightarrow{\mathbf{v}}_1$, $\overrightarrow{\mathbf{v}}_2, \ldots \overrightarrow{\mathbf{v}}_k$ that are all from $\mathbb{R}^n$ is called *linearly dependent* if and only if

$$c_1\overrightarrow{\mathbf{v}}_1 + c_2\overrightarrow{\mathbf{v}}_2 + \cdots + c_k\overrightarrow{\mathbf{v}}_k = \overrightarrow{\mathbf{0}}$$

has a *nonzero* solution for the c_i's; it is called *linearly independent* if the *only* solution is $c_1 = c_2 = \cdots = c_k = 0$.

(continued)

[†]This definition is equivalent to the one given in part (a): if one of the vectors has been expressed as a linear combination of the others, then we can subtract everything to one side to find a nonzero solution for $c_1\overrightarrow{\mathbf{v}}_1 + c_2\overrightarrow{\mathbf{v}}_2 + \cdots + c_k\overrightarrow{\mathbf{v}}_k = \overrightarrow{\mathbf{0}}$; conversely, if one of the c_i's in this equation is nonzero, then we can isolate its corresponding vector to express that vector as a linear combination of the others.

(continued) Therefore, we can determine if the given vectors form a linearly dependent set by making the zero vector a *linear combination* of these vectors and solving for the coefficients:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 3 & 2 & 0 \\ 3 & 4 & 2 & 0 \\ 0 & 5 & 4 & 0 \end{array}\right]$$

$$\begin{array}{c} R_2 - 4R_1 \\ R_3 - 3R_1 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -10 & 0 \\ 0 & -2 & -7 & 0 \\ 0 & 5 & 4 & 0 \end{array}\right]$$

$$\begin{array}{c} -\frac{1}{5}R_2 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -7 & 0 \\ 0 & 5 & 4 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - 2R_2 \\ R_3 + 2R_2 \\ R_4 - 5R_2 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

$$\begin{array}{c} -\frac{1}{3}R_3 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 + R_3 \\ R_2 - 2R_3 \\ R_4 + 6R_3 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that the *only* solution is $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

Hence, the given set of vectors is $\boxed{\text{linearly independent.}}$

Note: If you multiply zero by a nonzero number, the result is zero. If you interchange two zeros, you will still have two zeros. If you add (or subtract) zero times any number to (or from) zero, the result is [*surprise!*] zero. In other words, the elementary row operations will *not* change a column of zeros. We can therefore save a bit of writing when solving this type of problem in the future by *omitting* the column of zeros and *pretending* that it is there.

(c) $\begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$

Solution:

Again, we make the zero vector a linear combination of the given vectors and solve for the coefficients, but as per the suggestion from the previous page, we will omit the column of zeros from our augmented matrix and pretend that it's there:

$$\begin{bmatrix} 2 & -5 & 4 & 3 \\ -3 & 1 & 3 & 1 \\ 7 & 1 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow{R_1+R_2} \begin{bmatrix} -1 & -4 & 7 & 4 \\ -3 & 1 & 3 & 1 \\ 7 & 1 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow{-R_1} \begin{bmatrix} 1 & 4 & -7 & -4 \\ -3 & 1 & 3 & 1 \\ 7 & 1 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow[R_3-7R_1]{R_2+3R_1} \begin{bmatrix} 1 & 4 & -7 & -4 \\ 0 & 13 & -18 & -11 \\ 0 & -27 & 49 & 33 \end{bmatrix}$$

$$\xrightarrow{R_3+2R_2} \begin{bmatrix} 1 & 4 & -7 & -4 \\ 0 & 13 & -18 & -11 \\ 0 & -1 & 13 & 11 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 4 & -7 & -4 \\ 0 & -1 & 13 & 11 \\ 0 & 13 & -18 & -11 \end{bmatrix}$$

$$\xrightarrow{-R_2} \begin{bmatrix} 1 & 4 & -7 & -4 \\ 0 & 1 & -13 & -11 \\ 0 & 13 & -18 & -11 \end{bmatrix}$$

$$\xrightarrow[R_3-13R_2]{R_1-4R_2} \begin{bmatrix} 1 & 0 & 45 & 40 \\ 0 & 1 & -13 & -11 \\ 0 & 0 & 151 & 132 \end{bmatrix}$$

(continued)

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\begin{aligned}
&= \left[\begin{array}{rrrr} 1 & 0 & 45 & 40 \\ 0 & 1 & -13 & -11 \\ 0 & 0 & 151 & 132 \end{array}\right] \\
&\overset{\frac{1}{151}R_2}{\longrightarrow} \left[\begin{array}{rrrr} 1 & 0 & 45 & 6040/151 \\ 0 & 1 & -13 & -1661/151 \\ 0 & 0 & 1 & 132/151 \end{array}\right] \\
&\overset{\substack{R_1 - 45R_3 \\ R_2 + 13R_3}}{\longrightarrow} \left[\begin{array}{rrrr} 1 & 0 & 0 & 100/151 \\ 0 & 1 & 0 & 55/151 \\ 0 & 0 & 1 & 132/151 \end{array}\right].
\end{aligned}
$$

This matrix is in reduced row echelon form. Remembering that there is an invisible fifth column that contains all zeros, the associated system is

$$
\begin{aligned}
c_1 \qquad\qquad + \frac{100}{151}c_4 &= 0 \\
c_2 \qquad + \frac{55}{151}c_4 &= 0 \\
c_3 + \frac{132}{151}c_4 &= 0
\end{aligned}
$$

We see that c_4 is a free variable, so $c_4 = t$. Then, the solution set is

$$c_1 = -\frac{100}{151}t, \;\; c_2 = -\frac{55}{151}t, \;\; c_3 = -\frac{132}{151}t, \;\; c_4 = t.$$

Does this set include a *nonzero* solution? Yes! To find one, we just pick a nonzero number, like -151, and substitute it into t:

$$c_1 = 100, \;\; c_2 = 55, \;\; c_3 = 132, \;\; c_4 = -151.$$

Thus, the given set of vectors is $\boxed{\text{linearly dependent.}}$

To obtain a *dependence relation* among the vectors, we put our nonzero solution back into the vector equation from the definition of linear dependence:

$$\boxed{100\begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} + 55\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} + 132\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} - 151\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.}$$

(3) Canada's Health Guide recommends $625\mu g$ of vitamin A, $10\mu g$ of vitamin D, $12000\mu g$ of vitamin E, and 120 μg of vitamin K per day for the average male university student. In the University of Hard Knocks, the cafeteria offers three food items per day: special 1, special 2, and special 3. Here is the table of their vitamin content (all units are in μg):

	Special 1	Special 2	Special 3
Vitamin A	100	125	75
Vitamin D	1	2	2
Vitamin E	2000	2500	1000
Vitamin K	20	30	0

Can an average male university student here have a healthy diet? (Assume that a healthy diet means getting exactly the right amount of these four vitamins.)

Solution:

Let a, b, and c represent the number of Special 1, Special 2, and Special 3 (respectively) that a male student consumes in one day. Then, the daily total amount of vitamin A (in μg) that this student gets is $100a+125b+75c$. In order for his diet to be healthy, this student must get $625\mu g$ of vitamin A per day, so we need $100a+125b+75c=625$. Repeating this for the other three vitamins, we obtain a system of four linear equations in three unknowns:

$$\begin{cases} 100a + 125b + 75c = 625 \\ a + 2b + 2c = 10 \\ 2000a + 2500b + 1000c = 12000 \\ 20a + 30b = 120. \end{cases}$$

Let us solve this system in the usual way:

$$\left[\begin{array}{ccc|c} 100 & 125 & 75 & 625 \\ 1 & 2 & 2 & 10 \\ 2000 & 2500 & 1000 & 12000 \\ 20 & 30 & 0 & 120 \end{array}\right]$$

$$\begin{array}{c} R_1 \leftrightarrow R_2 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 100 & 125 & 75 & 625 \\ 2000 & 2500 & 1000 & 12000 \\ 20 & 30 & 0 & 120 \end{array}\right]$$

$$\begin{array}{c} R_2 - 100\,R_1 \\ R_3 - 2000\,R_1 \\ R_4 - 20\,R_1 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & -75 & -125 & -375 \\ 0 & -1500 & -3000 & -8000 \\ 0 & -10 & -40 & -80 \end{array}\right]$$

(continued)

(continued)

$$= \left[\begin{array}{rrr|r} 1 & 2 & 2 & 10 \\ 0 & -75 & -125 & -375 \\ 0 & -1500 & -3000 & -8000 \\ 0 & -10 & -40 & -80 \end{array}\right]$$

$$\overset{R_2 \leftrightarrow R_4}{\longrightarrow} \left[\begin{array}{rrr|r} 1 & 2 & 2 & 10 \\ 0 & -10 & -40 & -80 \\ 0 & -1500 & -3000 & -8000 \\ 0 & -75 & -125 & -375 \end{array}\right]$$

$$\overset{-\frac{1}{10} R_2}{\longrightarrow} \left[\begin{array}{rrr|r} 1 & 2 & 2 & 10 \\ 0 & 1 & 4 & 8 \\ 0 & -1500 & -3000 & -8000 \\ 0 & -75 & -125 & -375 \end{array}\right]$$

$$\overset{\substack{R_1 - 2R_2 \\ R_3 + 1500\,R_2 \\ R_4 + 75\,R_2}}{\longrightarrow} \left[\begin{array}{rrr|r} 1 & 0 & -6 & -6 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 3000 & 4000 \\ 0 & 0 & 175 & 225 \end{array}\right]$$

$$\overset{\substack{(1/3000)\,R_3 \\ (1/175)\,R_4}}{\longrightarrow} \left[\begin{array}{rrr|r} 1 & 0 & -6 & -6 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 1 & 4/3 \\ 0 & 0 & 1 & 9/7 \end{array}\right]$$

$$\overset{R_4 - R_3}{\longrightarrow} \left[\begin{array}{rrr|r} 1 & 0 & -6 & -6 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 1 & 4/3 \\ 0 & 0 & 0 & -1/21 \end{array}\right].$$

We see from the last row that this system has *no* solution.

Thus, an average male student here *cannot* have a healthy diet.

Well, what did you expect? It is the *University of Hard Knocks*, afterall! (^‿^)

(4) Prove that $\overrightarrow{\mathbf{u}}$, $\overrightarrow{\mathbf{v}}$, and $\overrightarrow{\mathbf{w}}$ are all in $\operatorname{span}\left(\overrightarrow{\mathbf{u}},\ \overrightarrow{\mathbf{u}}+\overrightarrow{\mathbf{v}},\ \overrightarrow{\mathbf{u}}+\overrightarrow{\mathbf{v}}+\overrightarrow{\mathbf{w}}\right)$.

Solution:

It suffices to show that each of the vectors $\overrightarrow{\mathbf{u}}$, $\overrightarrow{\mathbf{v}}$, and $\overrightarrow{\mathbf{w}}$ is a linear combination of $\overrightarrow{\mathbf{u}}$, $\overrightarrow{\mathbf{u}}+\overrightarrow{\mathbf{v}}$, and $\overrightarrow{\mathbf{u}}+\overrightarrow{\mathbf{v}}+\overrightarrow{\mathbf{w}}$:

$$\boxed{\overrightarrow{\mathbf{u}} \;=\; 1\overrightarrow{\mathbf{u}} + 0\left(\overrightarrow{\mathbf{u}}+\overrightarrow{\mathbf{v}}\right) + 0\left(\overrightarrow{\mathbf{u}}+\overrightarrow{\mathbf{v}}+\overrightarrow{\mathbf{w}}\right).}$$

$$\boxed{\overrightarrow{\mathbf{v}} \;=\; (-1)\overrightarrow{\mathbf{u}} + 1\left(\overrightarrow{\mathbf{u}}+\overrightarrow{\mathbf{v}}\right) + 0\left(\overrightarrow{\mathbf{u}}+\overrightarrow{\mathbf{v}}+\overrightarrow{\mathbf{w}}\right).}$$

$$\boxed{\overrightarrow{\mathbf{w}} \;=\; 0\overrightarrow{\mathbf{u}} + (-1)\left(\overrightarrow{\mathbf{u}}+\overrightarrow{\mathbf{v}}\right) + 1\left(\overrightarrow{\mathbf{u}}+\overrightarrow{\mathbf{v}}+\overrightarrow{\mathbf{w}}\right).}$$

(5) Show that regardless of what the vectors $\overrightarrow{\mathbf{u}}$, $\overrightarrow{\mathbf{v}}$, and $\overrightarrow{\mathbf{w}}$ are, the three vectors $\overrightarrow{\mathbf{u}}-\overrightarrow{\mathbf{v}}$, $\overrightarrow{\mathbf{v}}-\overrightarrow{\mathbf{w}}$, and $\overrightarrow{\mathbf{w}}-\overrightarrow{\mathbf{u}}$ are linearly dependent.

Solution:

It suffices to show that $c_1\left(\overrightarrow{\mathbf{u}}-\overrightarrow{\mathbf{v}}\right)+c_2\left(\overrightarrow{\mathbf{v}}-\overrightarrow{\mathbf{w}}\right)+c_3\left(\overrightarrow{\mathbf{w}}-\overrightarrow{\mathbf{u}}\right) = \overrightarrow{\mathbf{0}}$ has a nonzero solution:

$$\boxed{1\left(\overrightarrow{\mathbf{u}}-\overrightarrow{\mathbf{v}}\right)+1\left(\overrightarrow{\mathbf{v}}-\overrightarrow{\mathbf{w}}\right)+1\left(\overrightarrow{\mathbf{w}}-\overrightarrow{\mathbf{u}}\right) \;=\; \overrightarrow{\mathbf{0}}.}$$