MATH 1410
Solutions for Homework 7
Submitted Friday, March 15, 2013

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(1) Let $A = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $D = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$,

$E = \begin{bmatrix} 4 & 2 \end{bmatrix}$, and $F = \begin{bmatrix} -1 & 2 \end{bmatrix}$. Compute the indicated matrices (if possible).

(a) $A + 2D$

Solution:

The addition and scalar multiplication of matrices are done just like the addition and scalar multiplication of vectors:

$$A + 2D = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} + 2\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ -4 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & -6 \\ -5 & 7 \end{bmatrix}.}$$

(b) $B - C$

Solution:

A matrix with m rows and n columns is said to have *size* $m \times n$. The sum or difference of two matrices is *only* defined if they have the same size.

Since B is 2×3 and C is 3×2, their difference is $\boxed{\text{undefined.}}$

Alternately, we can say that $B - C$ $\boxed{\text{DNE (does not exist).}}$

(c) AB

Solution:

Each column in the product AB is a linear combination of the columns in A (ordered from left to right) whose coefficients are provided by the corresponding column in B (ordered from top to bottom). For example, the first column in AB is 4 times the first column in A plus 0 times the second column in A, while the second column in AB is -2 times the first column in A plus 2 times the second column in A, and so on:

$$AB = \left[\begin{array}{c|c} 3 & 0 \\ -1 & 5 \end{array}\right]\left[\begin{array}{c|c|c} 4 & -2 & 1 \\ 0 & 2 & 3 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 4\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 5 \end{bmatrix} & -2\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 5 \end{bmatrix} & 1\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 5 \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} \begin{bmatrix} 12 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} -6 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} & \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 15 \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} \begin{bmatrix} 12 \\ -4 \end{bmatrix} & \begin{bmatrix} -6 \\ 12 \end{bmatrix} & \begin{bmatrix} 3 \\ 14 \end{bmatrix} \end{array}\right] = \boxed{\begin{bmatrix} 12 & -6 & 3 \\ -4 & 12 & 14 \end{bmatrix}.}$$

(d) $D+BC$

Solution:

$$D+BC = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} + \left[\begin{array}{c|c|c} 4 & -2 & 1 \\ 0 & 2 & 3 \end{array}\right]\left[\begin{array}{c|c} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array}\right]$$

$$= \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} + \left[\begin{array}{c|c} 1\begin{bmatrix} 4 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 2 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 3 \end{bmatrix} & 2\begin{bmatrix} 4 \\ 0 \end{bmatrix} + 4\begin{bmatrix} -2 \\ 2 \end{bmatrix} + 6\begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{array}\right]$$

$$= \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} + \left[\begin{array}{c|c} \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -6 \\ 6 \end{bmatrix} + \begin{bmatrix} 5 \\ 15 \end{bmatrix} & \begin{bmatrix} 8 \\ 0 \end{bmatrix} + \begin{bmatrix} -8 \\ 8 \end{bmatrix} + \begin{bmatrix} 6 \\ 18 \end{bmatrix} \end{array}\right]$$

$$= \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} + \left[\begin{array}{c|c} \begin{bmatrix} 3 \\ 21 \end{bmatrix} & \begin{bmatrix} 6 \\ 26 \end{bmatrix} \end{array}\right] = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 21 & 26 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 3 & 3 \\ 19 & 27 \end{bmatrix}.}$$

(e) $E(AF)$

Solution:

Let us first calculate $AF = \left[\begin{array}{c|c} 3 & 0 \\ -1 & 5 \end{array}\right]\left[\begin{array}{c|c} -1 & 2 \end{array}\right]\ldots$

Hmmm...there are *two* column vectors in A, but there is only *one* coefficient in each column of F, so how do we create each linear combination? (O_o)

We don't! Instead, we say that AF is *undefined*, which means that $E(AF)$ is also $\boxed{\text{undefined.}}$

Let us generalize this: say that we want to compute the product AB, where A is an $m \times n$ matrix and B is a $p \times q$ matrix. Then, the number of column vectors in A is the number of *columns* in A, which is n, and the number of coefficients in any column of B is the number of *rows* in B, which is p; these need to match! If they do match, AB will have the same number of rows as A, which is m, since the columns of AB are linear combinations of the columns of A, and it will have the same number of columns as B, which is q, since each column in the second matrix provides the coefficients needed to create one column in the product.

Consequently, if A is $m \times n$ and B is $p \times q$, then their product AB is defined exactly when $n = p$; if it is defined, AB is $m \times q$.

(f) $F(DF)$

Solution:

D is 2×2 and F is 1×2; because the "inside numbers," 2 and 1, are different, DF is undefined.

Therefore, $F(DF)$ $\boxed{\text{DNE.}}$

(g) $AD - DA$

Solution:

Let us first calculate $AD = \left[\begin{array}{c|c} 3 & 0 \\ -1 & 5 \end{array}\right]\left[\begin{array}{c|c} 0 & -3 \\ -2 & 1 \end{array}\right]$

$$= \left[\begin{array}{c|c} 0\begin{bmatrix} 3 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 5 \end{bmatrix} & -3\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 5 \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 10 \end{bmatrix} & \begin{bmatrix} -9 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} 0 \\ -10 \end{bmatrix} & \begin{bmatrix} -9 \\ 8 \end{bmatrix} \end{array}\right] = \begin{bmatrix} 0 & -9 \\ -10 & 8 \end{bmatrix}.$$

(continued)

(continued) Next, we compute $DA = \left[\begin{array}{c|c} 0 & -3 \\ -2 & 1 \end{array}\right]\left[\begin{array}{c|c} 3 & 0 \\ -1 & 5 \end{array}\right]$

$$= \left[\begin{array}{c|c} 3\begin{bmatrix} 0 \\ -2 \end{bmatrix} - 1\begin{bmatrix} -3 \\ 1 \end{bmatrix} & 0\begin{bmatrix} 0 \\ -2 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 1 \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} 0 \\ -6 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -15 \\ 5 \end{bmatrix} \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} 3 \\ -7 \end{bmatrix} & \begin{bmatrix} -15 \\ 5 \end{bmatrix} \end{array}\right] = \begin{bmatrix} 3 & -15 \\ -7 & 5 \end{bmatrix}.$$

Finally, $AD - DA = \begin{bmatrix} 0 & -9 \\ -10 & 8 \end{bmatrix} - \begin{bmatrix} 3 & -15 \\ -7 & 5 \end{bmatrix} = \boxed{\begin{bmatrix} -3 & 6 \\ -7 & 3 \end{bmatrix}.}$

(2) Give an example of a nonzero 2×2 matrix A such that $A^2 = 0$.

Solution:

A formal approach would be to let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and compute $A^2 = AA$

$$= \left[\begin{array}{c|c} a & b \\ c & d \end{array}\right]\left[\begin{array}{c|c} a & b \\ c & d \end{array}\right] = \left[\begin{array}{c|c} a\begin{bmatrix} a \\ c \end{bmatrix} + c\begin{bmatrix} b \\ d \end{bmatrix} & b\begin{bmatrix} a \\ c \end{bmatrix} + d\begin{bmatrix} b \\ d \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} a^2 \\ ac \end{bmatrix} + \begin{bmatrix} cb \\ cd \end{bmatrix} & \begin{bmatrix} ba \\ bc \end{bmatrix} + \begin{bmatrix} db \\ d^2 \end{bmatrix} \end{array}\right] = \begin{bmatrix} a^2 + cb & ba + db \\ ac + cd & bc + d^2 \end{bmatrix}.$$

Setting each of these entries equal to 0, we get a system of *nonlinear* equations:

$$① : a^2 + bc = 0 \quad ② : b(a+d) = 0 \quad ③ : c(a+d) = 0 \quad ④ : d^2 + bc = 0 \ldots$$

YIKES! (O_o) We don't have the tools to solve such a system, so why we don't we just guess and check? If a and d are both zero, then ② and ③ are satisfied; moreover, ① and ④ will be satisfied if bc is also zero. Hence, if $a = b = d = 0$ and $c = 1$, A will be a nonzero matrix (since $c \neq 0$), and all four equations above will be satisfied, so $A^2 = 0$. Other choices can also work, so here are some possible answers:

$\boxed{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},}$ $\boxed{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},}$ $\boxed{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},}$ and $\boxed{\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.}$

(3) (a) Let $E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Calculate $E_1 A$, $E_2 A$, and $E_3 A$.

Solution:

$$E_1 A = \left[\begin{array}{c|c} 1 & 1 \\ 0 & 1 \end{array}\right] \left[\begin{array}{c|c} a & b \\ c & d \end{array}\right] = \left[\begin{array}{c|c} a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \end{bmatrix} & b\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \end{bmatrix} & \begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} d \\ d \end{bmatrix} \end{array}\right] = \boxed{\begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}.}$$

$$E_2 A = \left[\begin{array}{c|c} 0 & 1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{c|c} a & b \\ c & d \end{array}\right] = \left[\begin{array}{c|c} a\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \end{bmatrix} & b\begin{bmatrix} 0 \\ 1 \end{bmatrix} + d\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} 0 \\ a \end{bmatrix} + \begin{bmatrix} c \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ b \end{bmatrix} + \begin{bmatrix} d \\ 0 \end{bmatrix} \end{array}\right] = \boxed{\begin{bmatrix} c & d \\ a & b \end{bmatrix}.}$$

$$E_3 A = \left[\begin{array}{c|c} 1 & 0 \\ 0 & -3 \end{array}\right] \left[\begin{array}{c|c} a & b \\ c & d \end{array}\right] = \left[\begin{array}{c|c} a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ -3 \end{bmatrix} & b\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ -3 \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3c \end{bmatrix} & \begin{bmatrix} b \\ c \end{bmatrix} + \begin{bmatrix} 0 \\ -3d \end{bmatrix} \end{array}\right] = \boxed{\begin{bmatrix} a & b \\ -3c & -3d \end{bmatrix}.}$$

Food for thought: For each k from 1 to 3, consider what elementary row operation you would need to perform on the matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

to get E_k, then look at the transformation from A to $E_k A$. Neat, ey? This is the "mathemagic" of *elementary matrices* (to be discussed later).

(b) Find a 2×2 matrix $[E]$ so that $EA = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$;

that is, EA is the matrix that you get by applying $R_1 + 2R_2$ to A.

Solution:

Let's work *backwards:* $\left[\begin{array}{c|c} a+2c & b+2d \\ c & d \end{array}\right] = \left[\begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 2c \\ c \end{bmatrix} \;\middle|\; \begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} 2d \\ d \end{bmatrix} \right]$

$$= \left[a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 1 \end{bmatrix} \;\middle|\; b\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right] = \left[\begin{array}{c|c} 1 & 2 \\ 0 & 1 \end{array}\right]\left[\begin{array}{c|c} a & b \\ c & d \end{array}\right] = EA.$$

By observation, $E = \boxed{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.}$

Boy, working backwards is both difficult and confusing. I can't wait for the explanation of elementary matrices!

(4) Find a 3×3 matrix C so that

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(Hint: Find C column by column.)

Solution:

Let $C = \begin{bmatrix} \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \vec{\mathbf{v}}_3 \end{bmatrix}$, where each $\vec{\mathbf{v}}_i$ is a column vector from $\mathbb{R}^3$. Then,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \vec{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Because multiplying a vector on the right of a matrix creates a linear combination, we will find each $\vec{\mathbf{v}}_i$ by solving a *linear combination problem;* specifically, we will row reduce an augmented matrix whose three coefficient columns come from the matrix being multiplied by each $\vec{\mathbf{v}}_i$ and whose solution column is the vector on the right-hand side of the corresponding equation.

(continued)

(continued) To begin,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R_2-2R_1 \\ R_3-3R_1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & -1 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right].$$

Therefore, $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$. Next,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R_2-2R_1 \\ R_3-3R_1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

Thus, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Lastly,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_2-2R_1 \\ R_3-3R_1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right],$$

so $\vec{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Putting these back into C, we get $C = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 1 & 1 \end{bmatrix}$.

(5) Show that vectors $\overrightarrow{\mathbf{v}}_1$, $\overrightarrow{\mathbf{v}}_2$, $\overrightarrow{\mathbf{v}}_3$ in $\mathbb{R}^4$ can never span $\mathbb{R}^4$.

<u>Solution</u>:

Recall that we determine if a set of vectors from $\mathbb{R}^n$ span $\mathbb{R}^n$ by reducing the matrix whose column are these vectors: if the resulting reduced row echelon form matrix has a leading one in every row, then the given set spans $\mathbb{R}^n$; if some row does not have a leading one (in other words, there is a row of zeros), then the given set does not span $\mathbb{R}^n$.

Say that $\overrightarrow{\mathbf{v}}_1$, $\overrightarrow{\mathbf{v}}_2$, and $\overrightarrow{\mathbf{v}}_3$ have been given to us, and we create and row reduce the matrix whose columns are these vectors. Because these vectors are in $\mathbb{R}^4$, our matrix has four rows. Because we have been given three of them, our matrix has three columns. Hence, our matrix is 4×3.

When we find the reduced row echelon form of our matrix, how many leading ones will it have? Leading ones must be in different columns, so there can be at most three. However, our matrix has four rows, so at least one row will not get a leading one. Ergo, $\overrightarrow{\mathbf{v}}_1$, $\overrightarrow{\mathbf{v}}_2$, and $\overrightarrow{\mathbf{v}}_3$ can never span $\mathbb{R}^4$, as desired.