MATH 1410
Solutions for Homework 8
Submitted Thursday, March 28, 2013

For all of the questions below, let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 3 & 5 \end{bmatrix}$$

and let

$$\overrightarrow{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \overrightarrow{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \overrightarrow{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(1) Find [a] vector $\overrightarrow{\mathbf{x}}_1$ such that $A\overrightarrow{\mathbf{x}}_1 = \overrightarrow{\mathbf{e}}_1$.

Solution:

Hmmm...what *size* is $\overrightarrow{\mathbf{x}}_1$? Say that it is $m \times n$. Then, A is a 3×3 matrix, and the product $A\overrightarrow{\mathbf{x}}_1$ is defined, so $m = 3$. Moreover, $A\overrightarrow{\mathbf{x}}_1$ is $3 \times n$ and is equal to $\overrightarrow{\mathbf{e}}_1$, which is 3×1, so $n = 1$. Hence, $\overrightarrow{\mathbf{x}}_1$ is a 3×1 matrix, or a column vector from $\mathbb{R}^3$. The same argument shows that $\overrightarrow{\mathbf{x}}_2$ and $\overrightarrow{\mathbf{x}}_3$ are also column vectors from $\mathbb{R}^3$.

Accordingly, we let $\overrightarrow{\mathbf{x}}_1 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$.

Then, $A\overrightarrow{\mathbf{x}}_1 = \overrightarrow{\mathbf{e}}_1 \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\implies c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Recognizing this as a linear combination problem, we jump to row reducing the augmented matrix whose columns are the vectors.

(continued)

(continued)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -2 & 4 & 0 \\ 0 & 3 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 9 & 0 \\ 0 & 3 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 9 & 0 \\ 0 & 0 & -22 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{22}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 9 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2-9R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

We find that $c_1 = 1$, $c_2 = 0$, and $c_3 = 0$. Consequently, $\overrightarrow{\mathbf{x}}_1 = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$.

(2) Find [a] vector $\overrightarrow{\mathbf{x}}_2$ such that $A\overrightarrow{\mathbf{x}}_2 = \overrightarrow{\mathbf{e}}_2$.

Solution:

Like before, we let $\overrightarrow{\mathbf{x}}_2 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$.

Then, $A\overrightarrow{\mathbf{x}}_2 = \overrightarrow{\mathbf{e}}_2 \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 3 & 5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\implies c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(continued)

(continued) Again, we solve this linear combination problem by row reducing the augmented matrix whose columns are the vectors:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2 & 4 & 1 \\ 0 & 3 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 9 & 1 \\ 0 & 3 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 9 & 1 \\ 0 & 0 & -22 & -3 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{22}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 9 & 1 \\ 0 & 0 & 1 & 3/22 \end{array}\right]$$

$$\xrightarrow{R_2-9R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5/22 \\ 0 & 0 & 1 & 3/22 \end{array}\right].$$

Déjà vu! Anyways, we get $c_1 = 0$, $c_2 = -5/22$, and $c_3 = 3/22$, so $\overrightarrow{\mathbf{x}}_2 = \boxed{\begin{bmatrix} 0 \\ -5/22 \\ 3/22 \end{bmatrix}}.$

(3) Find [a] vector $\overrightarrow{\mathbf{x}}_3$ such that $A\overrightarrow{\mathbf{x}}_3 = \overrightarrow{\mathbf{e}}_3$.

Solution:

Like before, we let $\overrightarrow{\mathbf{x}}_3 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ ...waitaminute! Didn't I do this already? No? Okay, then.

Continuing, $A\overrightarrow{\mathbf{x}}_3 = \overrightarrow{\mathbf{e}}_3 \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\implies c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(continued)

(continued) Again, we solve this linear combination problem by row ARE YOU SURE I HAVEN'T DONE THIS ALREADY? Because it really feels like I have. No? Okay, let's row reduce, then:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 3 & 5 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 9 & 1 \\ 0 & 3 & 5 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 9 & 1 \\ 0 & 0 & -22 & -2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{22}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 9 & 1 \\ 0 & 0 & 1 & 1/11 \end{array}\right]$$

$$\xrightarrow{R_2-9R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2/11 \\ 0 & 0 & 1 & 1/11 \end{array}\right].$$

We find that $c_1 = 0$, $c_2 = 2/11$, and $c_3 =$ I'M SURE I'VE DONE THIS BEFORE! Except the last column. That looks different. BUT EVERYTHING ELSE IS THE SAME!

Anyhoo, $c_3 = 1/11$, so $\vec{\mathbf{x}}_3 = \boxed{\begin{bmatrix} 0 \\ 2/11 \\ 1/11 \end{bmatrix}}.$

Questions (1), (2), and (3), TAKE TWO!

Please excuse my outbursts above, but it felt like we just solved the same system of linear equations three times in a row. All that changed from one part to the next was the solution column. It would have been nice if we could have solved all three systems in one go.

Actually, we CAN solve all three systems simultaneously! The trick is to use *multiple* solution columns, like such:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 0 & 3 & 5 & 0 & 0 & 1 \end{array}\right].$$

(continued)

(continued) We then row reduce this matrix like we would any other 3×6 matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 0 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 9 & 0 & 1 & 1 \\ 0 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 9 & 0 & 1 & 1 \\ 0 & 0 & -22 & 0 & -3 & -2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{22}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 9 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 3/22 & 1/11 \end{array}\right]$$

$$\xrightarrow{R_2-9R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -5/22 & 2/11 \\ 0 & 0 & 1 & 0 & 3/22 & 1/11 \end{array}\right].$$

This is now in RREF, so how do we interpret it? By focusing on the column in which which we put the appropriate solution column. Say that we just want $\overrightarrow{\mathbf{x}}_2$, the vector satisfying $A\overrightarrow{\mathbf{x}}_2 = \overrightarrow{\mathbf{e}}_2$. To find it, we would solve the system of linear equations whose solution column is $\overrightarrow{\mathbf{e}}_2$, which is the *second* solution column in our augmented matrix above (column 5 overall). To solve that system alone, we *ignore* the first and third solution columns (columns 4 and 6 overall) in the RREF

$$\text{i.e.} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \blacksquare & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare & -5/22 & \blacksquare \\ 0 & 0 & 1 & \blacksquare & 3/22 & \blacksquare \end{array}\right]$$

and that find $c_1 = 0$, $c_2 = -5/22$, and $c_3 = 3/22$; in other words, $\overrightarrow{\mathbf{x}}_2 = \begin{bmatrix} 0 \\ -5/22 \\ 3/22 \end{bmatrix}$.

Doing this for all three solution columns allows us to answer questions (1), (2), *and* (3):

(1) $\overrightarrow{\mathbf{x}}_1 = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.}$ (2) $\overrightarrow{\mathbf{x}}_2 = \boxed{\begin{bmatrix} 0 \\ -5/22 \\ 3/22 \end{bmatrix}.}$ (3) $\overrightarrow{\mathbf{x}}_3 = \boxed{\begin{bmatrix} 0 \\ 2/11 \\ 1/11 \end{bmatrix}.}$

(4) Find A^{-1}.

Solution:

We are looking for a 3×3 matrix C such that $AC = CA = I_3$,[†] which requires that both $AC = I_3$ and $CA = I_3$. Let us solve the former first and deal with the latter latter…I mean, later.

Let $\overrightarrow{\mathbf{v}}_1$, $\overrightarrow{\mathbf{v}}_2$, and $\overrightarrow{\mathbf{v}}_3$ be the columns of C. Then, $AC = I_3$

$$\Longrightarrow A\left[\begin{array}{ccc} \overrightarrow{\mathbf{v}}_1 & \overrightarrow{\mathbf{v}}_2 & \overrightarrow{\mathbf{v}}_3 \end{array}\right] = \left[\begin{array}{ccc} \overrightarrow{\mathbf{e}}_1 & \overrightarrow{\mathbf{e}}_2 & \overrightarrow{\mathbf{e}}_3 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc} A\overrightarrow{\mathbf{v}}_1 & A\overrightarrow{\mathbf{v}}_2 & A\overrightarrow{\mathbf{v}}_3 \end{array}\right] = \left[\begin{array}{ccc} \overrightarrow{\mathbf{e}}_1 & \overrightarrow{\mathbf{e}}_2 & \overrightarrow{\mathbf{e}}_3 \end{array}\right]$$

$$\Longrightarrow A\overrightarrow{\mathbf{v}}_1 = \overrightarrow{\mathbf{e}}_1,\ A\overrightarrow{\mathbf{v}}_2 = \overrightarrow{\mathbf{e}}_2,\ \text{and } A\overrightarrow{\mathbf{v}}_3 = \overrightarrow{\mathbf{e}}_3.$$

Now we need to solve these three equa…WAITAMINUTE! We DID solve these three equations, in questions (1), (2), and (3)! So, the $\overrightarrow{\mathbf{x}}_i$'s that we found are apparently the columns of C

$$\text{i.e. } C = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -5/22 & 2/11 \\ 0 & 3/22 & 1/11 \end{array}\right].$$

We still don't know if this is the inverse of A, so let us compute the other product:

$$CA = \left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 0 & -5/22 & 2/11 \\ \hline 0 & 3/22 & 1/11 \end{array}\right]\left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 3 & 5 \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = I_3,$$

as required, so C is indeed the inverse of A; that is,

$$\boxed{A^{-1} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -5/22 & 2/11 \\ 0 & 3/22 & 1/11 \end{array}\right].}$$

[†]See the solution for (4)(e) on the Sample Second Midterm for an explanation.

(5) Let $\overrightarrow{\mathbf{b}} = b_1\overrightarrow{\mathbf{e}}_1 + b_2\overrightarrow{\mathbf{e}}_2 + b_3\overrightarrow{\mathbf{e}}_3 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Show that

$$A\big(b_1\overrightarrow{\mathbf{x}}_1 + b_2\overrightarrow{\mathbf{x}}_2 + b_3\overrightarrow{\mathbf{x}}_3\big) = \overrightarrow{\mathbf{b}}.$$

Solution:

We use the properties of vector and matrix algebra along with the equations that we solved in questions (1), (2), and (3):

$$A\big(b_1\overrightarrow{\mathbf{x}}_1 + b_2\overrightarrow{\mathbf{x}}_2 + b_3\overrightarrow{\mathbf{x}}_3\big) \;=\; A\big(b_1\overrightarrow{\mathbf{x}}_1\big) + A\big(b_2\overrightarrow{\mathbf{x}}_2\big) + A\big(b_3\overrightarrow{\mathbf{x}}_3\big) \;=\; b_1\big(A\overrightarrow{\mathbf{x}}_1\big) + b_2\big(A\overrightarrow{\mathbf{x}}_2\big) + b_3\big(A\overrightarrow{\mathbf{x}}_3\big)$$

$$=\; b_1\big(\overrightarrow{\mathbf{e}}_1\big) + b_2\big(\overrightarrow{\mathbf{e}}_2\big) + b_3\big(\overrightarrow{\mathbf{e}}_3\big) \;=\; b_1\overrightarrow{\mathbf{e}}_1 + b_2\overrightarrow{\mathbf{e}}_2 + b_3\overrightarrow{\mathbf{e}}_3 \;=\; \overrightarrow{\mathbf{b}},$$ as required.

(6) Let $\overrightarrow{\mathbf{x}}$ and $\overrightarrow{\mathbf{b}}$ be vectors such that $A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{b}}$. Show that $A^{-1}\overrightarrow{\mathbf{b}} = \overrightarrow{\mathbf{x}}$.

Solution:

Again, we use vector and matrix algebra:

$$A^{-1}\overrightarrow{\mathbf{b}} \;=\; A^{-1}\big(A\overrightarrow{\mathbf{x}}\big) \;=\; \big(A^{-1}A\big)\overrightarrow{\mathbf{x}} \;=\; (I)\overrightarrow{\mathbf{x}} \;=\; I\,\overrightarrow{\mathbf{x}} \;=\; \overrightarrow{\mathbf{x}},$$

as required.

(7) (**Bonus**) Let M be an $m \times n$ matrix. Show that there is [an] $n \times m$ matrix N such that $MN = I_m$ if and only if the column vectors of M span $\mathbb{R}^m$.

Solution:

Let P and Q represent statements. To prove "P if and only if Q," we actually have to do *two* proofs: "if P, then Q," which may be labelled as ($\Longrightarrow$), and "if Q, then P," which may be labelled as ($\Longleftarrow$).

($\Longleftarrow$) Assume that M is an $m \times n$ matrix whose column vectors span $\mathbb{R}^m$. Then, *every* vector in $\mathbb{R}^m$ may be expressed as a linear combination of the columns of M, *including* the columns of the $m \times m$ identity matrix. In particular, for each j from 1 to m inclusive, we can find coefficients for the n columns in M such that their linear combination yields $\overrightarrow{\mathbf{e}}_j$, the j-th column in I_m; let $\overrightarrow{\mathbf{x}}_j$ be the column vector from $\mathbb{R}^n$ whose entries are these coefficients (in the appropriate order).

Now let N be the $n \times m$ matrix whose columns are the $\overrightarrow{\mathbf{x}}_j$'s. Then,

$$MN = M\begin{bmatrix} \overrightarrow{\mathbf{x}}_1 & \overrightarrow{\mathbf{x}}_2 & \cdots & \overrightarrow{\mathbf{x}}_m \end{bmatrix} = \begin{bmatrix} M\overrightarrow{\mathbf{x}}_1 & M\overrightarrow{\mathbf{x}}_2 & \cdots & M\overrightarrow{\mathbf{x}}_m \end{bmatrix} = \begin{bmatrix} \overrightarrow{\mathbf{e}}_1 & \overrightarrow{\mathbf{e}}_2 & \cdots & \overrightarrow{\mathbf{e}}_m \end{bmatrix} = I_m,$$

as required.

($\Longrightarrow$) Assume that M is an $m \times n$ matrix and N is an $n \times m$ matrix such that $MN = I_m$. For each j from 1 to n inclusive, let $\overrightarrow{\mathbf{v}}_j$ be the j-th column in M. Next, let $\overrightarrow{\mathbf{b}}$ be any vector in $\mathbb{R}^m$

$$\text{i.e. } \overrightarrow{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix} = b_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + b_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + b_m\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = b_1\overrightarrow{\mathbf{e}}_1 + b_2\overrightarrow{\mathbf{e}}_2 + \cdots + b_m\overrightarrow{\mathbf{e}}_m.$$

By the definition of matrix multiplication, every column in MN is a linear combination of the columns of M. Since $MN = I_m$, every column in I_m is a linear combination of the columns of M; in other words, every $\overrightarrow{\mathbf{e}}_i$ may be expressed as a linear combination of the $\overrightarrow{\mathbf{v}}_j$'s. Consequently, we can replace each $\overrightarrow{\mathbf{e}}_i$ in the formula above by a linear combination of $\overrightarrow{\mathbf{v}}_j$'s to express $\overrightarrow{\mathbf{b}}$ as a linear combination of $\overrightarrow{\mathbf{v}}_j$'s!

Since $\overrightarrow{\mathbf{b}}$ was arbitrary, *every* vector in $\mathbb{R}^m$ may be expressed as a linear combination of the $\overrightarrow{\mathbf{v}}_j$'s; in other words, the columns vectors of M span $\mathbb{R}^m$, as required.