MATH 1410
Solutions for Homework 9
Submitted Friday, April 5[†], 2013

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(1) Calculate the rank of

$$\begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}.$$

([The *rank*] of a matrix is the number of pivot elements in the echelon form of the matrix.)

Solution:

Echelon form? This is a job for…**ROW REDUCTION!**

$$\begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{\substack{R_2 - 2R_1 \\ R_3 + R_1}} \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & -2 & -6 & -7 \\ 0 & 0 & 2 & 6 & 7 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_2} \begin{bmatrix} \mathbf{1} & -2 & 1 & 4 & 4 \\ 0 & 0 & -\mathbf{2} & -6 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is in row echelon form[‡] and has *two* pivot elements, so the rank of the given matrix is $\boxed{2.}$

[†]DOH! I apologize for the delay.

[‡]It is *not* in *reduced* row echelon form, but why do more work than necessary? $\begin{pmatrix} \sim & \circ \\ & \smile \end{pmatrix}$

(2) If A is a 3×5 matrix, explain why the columns of A must be linearly dependent.

Solution:

Let A be a 3×5 matrix. Clearly, the leading ones in the reduced row echelon form of a matrix must be in different rows. Since A has only three rows, its reduced row echelon form can have at most three leading ones. Since A has five columns, its reduced row echelon form has at least two columns that do not contain leading ones, which means that the system $\left[\begin{array}{c|c} A & \vec{\mathbf{0}} \end{array}\right]$ has at least two free variables. Letting these free variables be nonzero numbers yields a nonzero solution for this system. Lastly, the existence of this nonzero solution implies that the columns of A are linearly dependent, as required.

(3) Give an example of invertible matrices A and B, so that

$$(A+B)^{-1} \neq A^{-1}+B^{-1}.$$

Solution:

If we were to create random matrices, there is no guarantee that they would be *invertible,* let alone satisfy the inequality above. What's worse: we would not know if they were invertible or not until we row reduced them or calculated their determinants! (>_<)

So, let's *not* randomly select matrices; instead, let's use matrices that we *know* have inverses: identity matrices! Since $I_n I_n = I_n$, an identity matrix is its *own* inverse; that is, $I_n^{-1} = I_n$.

Accordingly, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then, $A^{-1}+B^{-1} = I_2^{-1}+I_2^{-1} = I_2+I_2 = 2I_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Next, $A+B = I_2+I_2 = 2I_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

I know what you're thinking: I'm going to attach an identity to this last matrix and use the Matrix Inversion Algorithm to find its inverse...NOPE! I will instead *multiply* the last two matrices:

$$(A+B)\left(A^{-1}+B^{-1}\right) = \begin{bmatrix} 2 & 0 \\ \hline 0 & 2 \end{bmatrix} \left[\begin{array}{c|c} 2 & 0 \\ 0 & 2 \end{array}\right] = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \neq I_2.$$

If the product of two matrices is *not* an identity matrix, then they are *not* inverses of each other. Consequently, $A^{-1}+B^{-1}$ is *not* the inverse of $A+B$; that is, $(A+B)^{-1} \neq A^{-1}+B^{-1}$, as required.

(4) Find three 2×2 matrices satisfying $A^2 = A$.

Solution:

If A was a number instead of a matrix, we'd be looking for numbers that don't change when we square them. Two such numbers are 0 and 1, so let's see if their 2×2 matrix "equivalents," $0_{2\times 2}$ and I_2, satisfy the given equation:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

They do! So, two such matrices are

$$A = \boxed{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \text{ and } A = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}.$$

Now imagine that these two matrices get married.* What will their kids look like? This, maybe?

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let's see if these kids inherited their parents' ability to satisfy the given equation:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

They did! So, two more such matrices are

$$A = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \text{ and } A = \boxed{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}.$$

*They enter into Holy Matrix-mony! [BA DUM TSSSH!] I'm here all week, folks!

(5) Let A be an $n \times m$ matrix. Show that if $A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$ and $A\overrightarrow{\mathbf{y}} = \overrightarrow{\mathbf{0}}$, then $A\overrightarrow{\mathbf{u}} = \overrightarrow{\mathbf{0}}$, where $\overrightarrow{\mathbf{u}}$ is any linear combination of $\overrightarrow{\mathbf{x}}$ and $\overrightarrow{\mathbf{y}}$.

<u>Solution</u>:

Let $\overrightarrow{\mathbf{x}}$ and $\overrightarrow{\mathbf{y}}$ be any vectors satisfying $A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$ and $A\overrightarrow{\mathbf{y}} = \overrightarrow{\mathbf{0}}$, and let $\overrightarrow{\mathbf{u}}$ be any linear combination of $\overrightarrow{\mathbf{x}}$ and $\overrightarrow{\mathbf{y}}$. Then, by the definition of linear combination, we can find number c_1 and c_2 such that

$$c_1\overrightarrow{\mathbf{x}} + c_2\overrightarrow{\mathbf{y}} = \overrightarrow{\mathbf{u}}.$$

We then use the properties of matrix algebra:

$$A\overrightarrow{\mathbf{u}} = A\left(c_1\overrightarrow{\mathbf{x}} + c_2\overrightarrow{\mathbf{y}}\right) = A\left(c_1\overrightarrow{\mathbf{x}}\right) + A\left(c_2\overrightarrow{\mathbf{y}}\right)$$

$$= c_1\left(A\overrightarrow{\mathbf{x}}\right) + c_2\left(A\overrightarrow{\mathbf{y}}\right) = c_1\left(\overrightarrow{\mathbf{0}}\right) + c_2\left(\overrightarrow{\mathbf{0}}\right) = \overrightarrow{\mathbf{0}} + \overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{0}}, \text{ as required.}$$