Sample Final - Answers

(1) The RREF of the matrix corresponding to this system of equation is

$$\begin{bmatrix} 1 & 0 & 0 & -35/11 \\ 0 & 1 & 0 & 3/11 \\ 0 & 0 & 1 & 29/11 \end{bmatrix}$$

which shows $a = -35/11$, $b = 3/11$, and $c = 29/11$.

(2) A REF of the matrix corresponding to this system of equation is

$$\begin{bmatrix} 1 & -1 & -k & -5 \\ 0 & 5 & 2k+1 & 15 \\ 0 & 0 & -4k+13 & 10 \end{bmatrix}$$

This matrix corresponds to a system with no solution when $-4k + 13 = 0$ (in this case, the last equation says $0 = 10$). Therefore, the system has no solution when $k = 13/4$.

(3) Using projection formula we get

$$\frac{v \cdot w}{w \cdot w} w = \frac{7}{21} w = \frac{w}{3}.$$

(4) (a) The RREF of matrix

$$\begin{bmatrix} 4 & -2 & 1 & 0 \\ 4 & -2 & 0 & 1 \\ -2 & -1 & 4 & -2 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 0 & -3/8 \\ 0 & 1 & 0 & -5/4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Since there are no zero rows in the REF of this matrix, these vectors span $\mathbb{R}^3$. Since the last column is not a pivot column we get that they are linearly dependent.

(b) The RREF of the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 0 & 3/7 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row, they will not span $\mathbb{R}^4$ but since all columns are pivot columns, they are linearly independent.

(5) See midterm 2.

(6) (a) We get

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

corresponds to $R_2 + 3R_1$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

corresponds to $2R_2$, and

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

corresponds to $R_1 \leftrightarrow R_3$.

(b) Using row expansion we get $\det(E_1) = 1$, $\det(E_2) = 2$, and $\det(E_3) = -1$.

(a) We can use row reduction to compute A^{-1}. Doing so we get

$$A^{-1} = \begin{bmatrix} 5 & -1/2 & -3 \\ -3 & 1/2 & 2 \\ -2 & 1/2 & 1 \end{bmatrix}.$$

(b) Note that $A^{-1}(AB) = B$. Therefore we get

$$B = A^{-1}(AB) = \begin{bmatrix} 5 & -1/2 & -3 \\ -3 & 1/2 & 2 \\ -2 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ & 01 & 1 \end{bmatrix} = \begin{bmatrix} 9/2 & 2 & -7/2 \\ -5/2 & -1 & 5/2 \\ -3/2 & -1 & 3/2 \end{bmatrix}$$

(7) Row exapsion along the second row is the most reasonable thing, and in that case we get $\det(A) = 15$.

(8) (a) Note that $\det(A - \lambda I_2) = \lambda^2 - 2\lambda - 8$. The roots of this quadratic are $\lambda_1 = 4$ and $\lambda_2 = -2$. Hence the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -2$.

(b) To find the eigenvector corresponding to $\lambda_1 = 4$ we need to find vector v_1 such that $(A - 4I)v_1 = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} v_1 = 0$. We can use $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
To find eigenvector corresponding to $\lambda_2 = -2$ we need to find vector v_2 such that $(A + 2I)v_2 = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} v_2 = 0$. $v_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}$ works.

(c) Note that

$$\frac{1}{2}v_1 + \frac{1}{2}v_2 = e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\frac{1}{2}v_1 - \frac{1}{2}v_2 = e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore we get

$$A^5 e_1 = A^5(\frac{1}{2}v_1 + \frac{1}{2}v_2) = \frac{4^5}{2}v_1 + \frac{(-2)^5}{2}v_2 = \begin{bmatrix} 512 \\ 512 \end{bmatrix} + \begin{bmatrix} -16 \\ 16 \end{bmatrix} = \begin{bmatrix} 496 \\ 528 \end{bmatrix}$$

and

$$A^5 e_2 = A^5(\frac{1}{2}v_1 - \frac{1}{2}v_2) = \frac{4^5}{2}v_1 - \frac{(-2)^5}{2}v_2 = \begin{bmatrix} 512 \\ 512 \end{bmatrix} + \begin{bmatrix} 16 \\ -16 \end{bmatrix} = \begin{bmatrix} 528 \\ 496 \end{bmatrix}$$

. Therefore we get

$$A^5 = \begin{bmatrix} 496 & 528 \\ 528 & 496 \end{bmatrix}.$$