

Sample, ugly row reduction.

This is a sample of row reduction when numbers don't work out nicely.

Example 1. Row reduce the matrix

$$\begin{bmatrix} 9 & -79 & -1 & 3 & -13 \\ 1 & 3 & 22 & 0 & 0 \\ -3 & -1 & -4 & -1 & 0 \\ 5 & -1 & 1 & -3 & 1 \end{bmatrix}$$

to reduce row echelon form.

Here we go. We start by moving second row to the first row (1 is the best pivot we can find).

$$\begin{aligned} \begin{bmatrix} 9 & -79 & -1 & 3 & -13 \\ 1 & 3 & 22 & 0 & 0 \\ -3 & -1 & -4 & -1 & 0 \\ 5 & -1 & 1 & -3 & 1 \end{bmatrix} &\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 22 & 0 & 0 \\ 9 & -79 & -1 & 3 & -13 \\ -3 & -1 & -4 & -1 & 0 \\ 5 & -1 & 1 & -3 & 1 \end{bmatrix} \\ &\xrightarrow{\substack{R_2 - 9R_1 \\ R_3 + 3R_1 \\ R_4 - 5R_1}} \begin{bmatrix} 1 & 3 & 22 & 0 & 0 \\ 0 & -106 & -199 & 3 & -13 \\ 0 & 8 & 62 & -1 & 0 \\ 0 & -16 & -109 & -3 & 1 \end{bmatrix} \\ &\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & 22 & 0 & 0 \\ 0 & 8 & 62 & -1 & 0 \\ 0 & -106 & -199 & 3 & -13 \\ 0 & -16 & -109 & -3 & 1 \end{bmatrix} \\ &\xrightarrow{\substack{R_3 + \frac{53}{4}R_2 \\ R_4 + 2R_2}} \begin{bmatrix} 1 & 3 & 22 & 0 & 0 \\ 0 & 8 & 62 & -1 & 0 \\ 0 & 0 & 1245/2 & -41/4 & -13 \\ 0 & 0 & 15 & -5 & 1 \end{bmatrix} \\ &\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 3 & 22 & 0 & 0 \\ 0 & 8 & 62 & -1 & 0 \\ 0 & 0 & 15 & -5 & 1 \\ 0 & 0 & 1245/2 & -41/4 & -13 \end{bmatrix} \\ &\xrightarrow{R_4 - \frac{83}{2}R_3} \begin{bmatrix} 1 & 3 & 22 & 0 & 0 \\ 0 & 8 & 62 & -1 & 0 \\ 0 & 0 & 15 & -5 & 1 \\ 0 & 0 & 0 & 789/4 & -109/2 \end{bmatrix} \end{aligned}$$

This brings it to Row Echelon Form. We can now reduce further, however, few notes about the swaps that I did. I usually like to have my pivot to be as simple as possible. So, between 8, 16, and 106, I would rather have 8 to be the pivot. Similarly, between 15 and 1245/2, I think 15 is a better pivot to have. Also, I tried to simplify the fractions as soon as possible. So, 106/8 is replaced by 53/4, and 16/8 is replaced by 2 before any more action takes place.

In any case, let's continue on reducing the matrix further.

$$\begin{aligned}
 & \begin{bmatrix} 1 & 3 & 22 & 0 & 0 \\ 0 & 8 & 62 & -1 & 0 \\ 0 & 0 & 15 & -5 & 1 \\ 0 & 0 & 0 & 789/4 & -109/2 \end{bmatrix} \xrightarrow{\frac{4}{789}R_4} \begin{bmatrix} 1 & 3 & 22 & 0 & 0 \\ 0 & 8 & 62 & -1 & 0 \\ 0 & 0 & 15 & -5 & 1 \\ 0 & 0 & 0 & 1 & -218/789 \end{bmatrix} \\
 & \xrightarrow{\begin{matrix} R_3+5R_4 \\ R_2+R_4 \end{matrix}} \begin{bmatrix} 1 & 3 & 22 & 0 & 0 \\ 0 & 8 & 62 & 0 & 218/789 \\ 0 & 0 & 15 & 0 & -301/789 \\ 0 & 0 & 0 & 1 & -218/789 \end{bmatrix} \\
 & \xrightarrow{\frac{1}{15}R_3} \begin{bmatrix} 1 & 3 & 22 & 0 & 0 \\ 0 & 8 & 62 & 0 & 218/789 \\ 0 & 0 & 1 & 0 & -301/11835 \\ 0 & 0 & 0 & 1 & -218/789 \end{bmatrix} \\
 & \xrightarrow{\begin{matrix} R_2-62R_3 \\ R_1-22R_3 \end{matrix}} \begin{bmatrix} 1 & 3 & 0 & 0 & 6622/11835 \\ 0 & 8 & 0 & 0 & 15392/11835 \\ 0 & 0 & 1 & 0 & -301/11835 \\ 0 & 0 & 0 & 1 & -218/789 \end{bmatrix} \\
 & \xrightarrow{\frac{1}{8}R_2} \begin{bmatrix} 1 & 3 & 0 & 0 & 6622/11835 \\ 0 & 1 & 0 & 0 & 1924/11835 \\ 0 & 0 & 1 & 0 & -301/11835 \\ 0 & 0 & 0 & 1 & -218/789 \end{bmatrix} \\
 & \xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 170/2367 \\ 0 & 1 & 0 & 0 & 1924/11835 \\ 0 & 0 & 1 & 0 & -301/11835 \\ 0 & 0 & 0 & 1 & -218/789 \end{bmatrix}
 \end{aligned}$$

So, we get the RREF for the matrix as it was desired. One quick note... I did use a computer to do most of arithmetic involved in this problem. So, for example, multiplying a row by $83/2$, you probably want to do a calculator for it. However, I used exact values the whole time. So, when I had to do $1 + 5 \cdot \frac{-218}{789}$ in the first step trying to go from REF to RREF I did

$$1 + 5 \cdot \frac{-218}{789} = \frac{789 - 5 \times 218}{789},$$

and I used my laptop to compute the numerator.