MATH 1410
Solutions for the Sample Midterm

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(1) Find the roots of the following polynomials:

(a) $(x-1)\left(x^2+5x+6\right)$

Solution:

Let a and b be numbers. Then,

$$(x+a)(x+b) \;=\; x^2+ax+bx+ab \;=\; x^2+(a+b)x+ab.$$

This formula can help us *factor* quadratic (second degree) polynomials! Say that we have a quadratic polynomial that is *monic* (the coefficient of its x^2 is one): if we can find numbers a and b whose sum is the coefficient of x and whose product is the constant term, then this polynomial may be factored as $(x+a)(x+b)$.

Consider x^2+5x+6: it is second degree, and the coefficient of its x^2 is 1, so we look for numbers a and b whose sum is 5 and whose product is 6. Quickly, we find 2 and 3, so

$$(x-1)\left(x^2+5x+6\right) \;=\; (x-1)(x+2)(x+3).$$

Then, the roots of this polynomial are $\boxed{1,\ -2,\ \text{and}\ -3.}$

(b) $(x-4)^2-3$

Solution:

We use the Difference of Squares formula: $(x-4)^2-3 \;=\; (x-4)^2-\left(\sqrt{3}\right)^2$

$$= \left((x-4)-\sqrt{3}\right)\left((x-4)+\sqrt{3}\right) \;=\; \left(x-\left(4+\sqrt{3}\right)\right)\left(x-\left(4-\sqrt{3}\right)\right).$$

So, the roots of this polynomial are $\boxed{4+\sqrt{3}\ \text{and}\ 4-\sqrt{3}.}$

(2) Solve the following systems of equations:

(a)

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 7 \\ 2x & + & y & - & 3z & = & 7 \\ x & + & 2y & - & 2z & = & 6 \end{array}$$

Solution:

We first number the equations:

$$\begin{array}{rrcrcrcr} ①: & x & + & y & + & z & = & 7 \\ ②: & 2x & + & y & - & 3z & = & 7 \\ ③: & x & + & 2y & - & 2z & = & 6 \end{array}$$

We then eliminate the x:

$$② - 2 \times ①: \ (2x+y-3z) - 2(x+y+z) = 7 - 2(7)$$

$$\implies \ 2x+y-3z-2x-2y-2z = 7-14 \implies -y-5z = -7 \ : ④$$

$$③ - ①: \ (x+2y-2z) - (x+y+z) = 6-7$$

$$\implies \ x+2y-2z-x-y-z = -1 \implies y-3z = -1 \ : ⑤$$

Next, we eliminate the y:

$$④ + ⑤: \ (-y-5z) + (y-3z) = -7 + (-1) \implies -8z = -8 \implies z = 1.$$

Now we can solve for y:

$$⑤: \ y - 3z = -1 \implies y - 3(1) = -1 \implies y - 3 = -1 \implies y = 3 - 1 = 2.$$

Finally, we solve for x:

$$①: \ x+y+z = 7 \implies x+2+1 = 7 \implies x+3 = 7 \implies x = 7-3 = 4.$$

Therefore, the given system has one solution, $(x, y, z) = \boxed{(4,\ 2,\ 1).}$

(b)

$$\begin{array}{rcrcrcr} 2a & + & 3b & + & 4c & = & 5 \\ 3a & + & 2b & + & 3c & = & 2 \\ & & 5b & + & 6c & = & 11 \end{array}$$

<u>Solution</u>:

Again, we number the equations:

$$\begin{array}{rrcrcrcr} ①: & 2a & + & 3b & + & 4c & = & 5 \\ ②: & 3a & + & 2b & + & 3c & = & 2 \\ ③: & & & 5b & + & 6c & = & 11 \end{array}$$

We then eliminate the a:

$3 \times ① - 2 \times ②:\ 3(2a+3b+4c) - 2(3a+2b+3c) = 3(5) - 2(2)$

$\implies\ 6a+9b+12c-6a-4b-6c = 15-4 \implies 5b+6c = 11\ldots$

Hmmm...this is the same as ③! One of these equations is redundant, leaving us with one equation to solve for two variables. In this situation, one of the variables is free to take any value.

Accordingly, let $c = t$. Then,

③: $5b+6c = 11 \implies 5b+6t = 11 \implies 5b = 11-6t \implies b = 2.2-1.2t.$

We now substitute both of these expressions into an equation containing a:

①: $2a+3b+4c = 5 \implies 2a+3(2.2-1.2t)+4t = 5 \implies 2a+6.6-3.6t+4t = 5$

$\implies\ 2a+0.4t = 5-6.6 \implies 2a = -1.6-0.4t \implies a = -0.8-0.2t.$

Thus, the given system has infinitely many solutions of the form

$$(a, b, c) = \boxed{\left(\ -0.8-0.2t\,,\ \ 2.2-1.2t\,,\ \ t\ \right).}$$

(c)

$$\begin{array}{rcrcrcrcl} a & + & b & + & c & + & d & = & 9 \\ 2a & + & 3b & + & c & + & d & = & 5 \\ & & & & a & + & 2b & = & 1 \end{array}$$

Solution:

We number the equations, as usual, and also line up the variables in the third equation:

$$\begin{array}{rrcrcrcrcl} ①: & a & + & b & + & c & + & d & = & 9 \\ ②: & 2a & + & 3b & + & c & + & d & = & 5 \\ ③: & a & + & 2b & & & & & = & 1 \end{array}$$

We then eliminate the a:

$② - 2 \times ①: \ (2a+3b+c+d) - 2(a+b+c+d) = 5 - 2(9)$

$\implies \ 2a+3b+c+d-2a-2b-2c-2d = 5-18 \implies b-c-d = -13 \ : ④$

$③ - ①: \ (a+2b) - (a+b+c+d) = 1-9$

$\implies \ a+2b-a-b-c-d = -8 \implies b-c-d = -8 \ : ⑤\ldots$

Hmmm…④ and ⑤ contradict each other: no matter what a, b, c, and d are, the expression $b-c-d$ cannot be equal to -13 and -8 at the same time!

We therefore conclude that the given system has $\boxed{\textit{no}\text{ solution.}}$

(3) Give an example of a 3-equations and 3-unknowns system of linear equations that has no solutions.

Solution:

We begin by creating an arbitrary equation with three variables: $a+2b+3c = 4$.

We then create a second equation that contradicts the first: $a+2b+3c = 5$.

Since the system now has no solutions, the third equation can be any equation with the same three variables, like $3a+5b+7c = 9$.

So, a system with the desired properties is $\boxed{\left\{\begin{array}{rcrcrcl} a & + & 2b & + & 3c & = & 4 \\ a & + & 2b & + & 3c & = & 5 \\ 3a & + & 5b & + & 7c & = & 9. \end{array}\right.}$

(4) Let $\overrightarrow{\mathbf{u}} = [\,1,\,2,\,-3\,]$ and $\overrightarrow{\mathbf{v}} = [\,-3,\,5,\,2\,]$.

(a) Calculate $\|\,\overrightarrow{\mathbf{u}}\,\|$

Solution:

$$\|\,\overrightarrow{\mathbf{u}}\,\| \;=\; \left\|\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}\right\| \;=\; \sqrt{1^2+2^2+(-3)^2} \;=\; \sqrt{1+4+9} \;=\; \boxed{\sqrt{14}.}$$

(b) Calculate $\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}}$

Solution:

$$\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} \;=\; \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \;=\; (1)(-3)+(2)(5)+(-3)(2) \;=\; -3+10-6 \;=\; \boxed{1.}$$

(c) Calculate the projection of vector $\overrightarrow{\mathbf{u}}$ onto vector $\overrightarrow{\mathbf{v}}$.

Solution:

Let us first calculate $\overrightarrow{\mathbf{v}} \cdot \overrightarrow{\mathbf{v}} \;=\; \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \;=\; (-3)^2+5^2+2^2 \;=\; 9+25+4 \;=\; 38.$

Then, $\operatorname{proj}_{\overrightarrow{\mathbf{v}}}(\overrightarrow{\mathbf{u}}) \;=\; \left(\dfrac{\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}}}{\overrightarrow{\mathbf{v}} \cdot \overrightarrow{\mathbf{v}}}\right)\overrightarrow{\mathbf{v}} \;=\; \left(\dfrac{1}{38}\right)\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \;=\; \boxed{\begin{bmatrix} -3/38 \\ 5/38 \\ 1/19 \end{bmatrix}.}$

(d) Show that the vector $[\,-1,\,9,\,-4\,]$ is a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$.

Solution:

The given vector (let's call it $\overrightarrow{\mathbf{w}}$) is a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ if we can find numbers c_1 and c_2 such that $c_1\overrightarrow{\mathbf{u}} + c_2\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{w}}$

$$\Longrightarrow\; c_1\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ -4 \end{bmatrix} \;\Longrightarrow\; \begin{bmatrix} c_1 \\ 2c_1 \\ -3c_1 \end{bmatrix} + \begin{bmatrix} -3c_2 \\ 5c_2 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ -4 \end{bmatrix}$$

(continued)

$$\text{(continued)} \implies \begin{bmatrix} c_1 - 3c_2 \\ 2c_1 + 5c_2 \\ -3c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ -4 \end{bmatrix} \implies \begin{cases} ①: & c_1 - 3c_2 = -1 \\ ②: & 2c_1 + 5c_2 = 9 \\ ③: & -3c_1 + 2c_2 = 4. \end{cases}$$

We now solve this system. To begin, let us eliminate c_1:

$$② - 2 \times ①:\ (2c_1 + 5c_2) - 2(c_1 - 3c_2) = 9 - 2(-1)$$

$$\implies 2c_1 + 5c_2 - 2c_1 + 6c_2 = 9 + 2 \implies 11c_2 = 11 \implies c_2 = 1.$$

We can now use this to find a value for c_1:

$$c_1 - 3c_2 = -1 \implies c_1 - 3(1) = -1 \implies c_1 - 3 = -1 \implies c_1 = 3 - 1 = 2.$$

Lastly, let us check that these values work:

$$2\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2-3 \\ 4+5 \\ -6+2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ -4 \end{bmatrix} \overset{\surd}{=} \overrightarrow{\mathbf{w}}.$$

So, the given vector *is* a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$, as expected.

(5) Give an example of three vectors $\overrightarrow{\mathbf{u}}$, $\overrightarrow{\mathbf{v}}$, and $\overrightarrow{\mathbf{w}}$ such that $\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}}$, but $\overrightarrow{\mathbf{v}} \neq \overrightarrow{\mathbf{w}}$.

Solution:

First, we select a vector space, like $\mathbb{R}^3$. The easiest way to find three such vectors is to let $\overrightarrow{\mathbf{v}}$ and $\overrightarrow{\mathbf{w}}$ be different vectors from $\mathbb{R}^3$ (say, $\overrightarrow{\mathbf{v}} = [1, 2, 3]$ and $\overrightarrow{\mathbf{w}} = [4, 5, 6]$) and let $\overrightarrow{\mathbf{u}}$ be the *zero vector* in $\mathbb{R}^3$ i.e. $\overrightarrow{\mathbf{v}} = [0, 0, 0]$. Then, $\overrightarrow{\mathbf{v}} \neq \overrightarrow{\mathbf{w}}$, while

$$\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0 + 0 + 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}}, \text{ as desired.}$$

(6) Show that $\left(\overrightarrow{\mathbf{u}} - \overrightarrow{\mathbf{v}}\right) \cdot \left(\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}}\right) = \| \overrightarrow{\mathbf{u}} \|^2 - \| \overrightarrow{\mathbf{v}} \|^2$.

Solution:

$$\text{LHS} = \left(\overrightarrow{\mathbf{u}} - \overrightarrow{\mathbf{v}}\right) \cdot \left(\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}}\right) = \left(\overrightarrow{\mathbf{u}} - \overrightarrow{\mathbf{v}}\right) \cdot \overrightarrow{\mathbf{u}} + \left(\overrightarrow{\mathbf{u}} - \overrightarrow{\mathbf{v}}\right) \cdot \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{u}} - \overrightarrow{\mathbf{v}} \cdot \overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} - \overrightarrow{\mathbf{v}} \cdot \overrightarrow{\mathbf{v}}$$

$$= \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{u}} - \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} - \overrightarrow{\mathbf{v}} \cdot \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{u}} - \overrightarrow{\mathbf{v}} \cdot \overrightarrow{\mathbf{v}} = \| \overrightarrow{\mathbf{u}} \|^2 - \| \overrightarrow{\mathbf{v}} \|^2 \overset{\surd}{=} \text{RHS.} \quad \blacksquare$$