Sample Second Midterm

(1) Solve the following systems of equations.

(a)

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 9, \\ 2x_2 - x_2 + x_3 &= 0, \\ 4x_1 - x_2 + x_3 &= 4 \end{aligned}$$

(b)

$$\begin{aligned} a + b + c + d &= 4 \\ a + 2b + 3c + 4d &= 10 \\ a + 3b + 6c + 10d &= 20 \\ a + 4b + 10c + 20d &= 35 \end{aligned}$$

(2) Row reduce the following matrices to reduced row echelon form, and circle the pivot elements.

(a)

$$\begin{bmatrix} 4 & -2 & 1 & 0 \\ 4 & -2 & 0 & 1 \\ -2 & -1 & 4 & -2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & -2 \\ 0 & 3/7 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(3) (a) Decide if the vectors

$$\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

span $\mathbb{R}^3$, and if they are linearly dependent or independent.

(b) Decide if the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3/7 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

span $\mathbb{R}^4$, and if they are linearly dependent or independent.

(4) Let

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 0 & 3 & -1 \end{bmatrix}, B = \begin{bmatrix} -2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}.$$

Calculate each of the following matrices if possible. If not, say undefined.

(a) $A + B$
(b) $AB + C$
(c) $BA + C$
(d) CAB

(e) $B + A^{-1}$
(f) C^{-1}

(5) Let $A = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, and assume that

$$A^5 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Solve for x.

(6) Find 2×2 matrices A and B so that $AB \neq BA$.

Answers

(1) For these questions, we can set up the augmented matrix and row reduce the matrices to reduce row echelon form. From the RREF of the matrix, we can read of the solutions.

(a) The RREF of the augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Therefore $x_1 = 2$, $x_2 = 5$ and $x_3 = 1$.

(b) The RREF of the augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(2) (a) The RREF of matrix

$$\begin{pmatrix} 4 & -2 & 1 & 0 \\ 4 & -2 & 0 & 1 \\ -2 & -1 & 4 & -1 \end{pmatrix}$$

is

$$\begin{pmatrix} 1 & 0 & 0 & -5/8 \\ 0 & 1 & 0 & -7/4 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(All the positive ones in the above matrix should be circled.)

(b) The RREF of matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 0 & 3/7 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Again, all the positive 1s should be circled.

(3) For vectors $v_1, \cdots, v_k$ we set up the matrix $[v_1 v_2 \cdots v_k]$ and compute a REF of this matrix. If has no zero rows, then it spans everything. If it has no free variables, the vectors are linearly independent.

As it happened, I did this question before question 2, so I didn't have the RREF of the matrices below available for me, so I computed the REF again.

(a) A REF of matrix

$$\begin{pmatrix} 4 & -2 & 1 & 0 \\ 4 & -2 & 0 & 1 \\ -2 & -1 & 4 & -1 \end{pmatrix}$$

is

$$\begin{pmatrix} -2 & -1 & 4 & -1 \\ 0 & -4 & 9 & -2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

(your matrix might be different, but it should have the same shape.) Notice that this matrix has no zero rows, therefore the four vectors in question span $\mathbb{R}^3$. However, the last column is not a pivot column, therefore these vectors are linearly dependent.

(b) A REF of matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 0 & 3/7 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there is a zero row at the bottom, we get that these three vectors can't span $\mathbb{R}^4$, however, since every column is a pivot column we get that they are linearly independent.

(4) (a) Undefined (sizes are different)

(b) $AB = \begin{pmatrix} 1 & 13 \\ 3 & 1 \end{pmatrix}$, and hence $AB + C = \begin{pmatrix} 3 & 13 \\ 4 & 3 \end{pmatrix}$.

(c) BA will be a three by three matrix, so $BA + C$ is undefined.

(d) We've already calculately AB. Therefore $CAB = \begin{pmatrix} 2 & 26 \\ 7 & 15 \end{pmatrix}$.

(e) A^{-1} is undefined, since A is not a square matrix.

(f) We solve for C^{-1}. We want a 2×2 matrix such that

$$\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

That is

$$\begin{pmatrix} 2a & 2b \\ a + 2c & b + 2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Solving the system of equations we get

$$a = 1/2,\ b = 0,\ c = -1/4,\ d = 1/2.$$

Therefore

$$C^{-1} = \begin{pmatrix} 1/2 & 0 \\ -1/4 & 1/2 \end{pmatrix}.$$

(5) Simple computation shows that

$$A^5 = \begin{pmatrix} 1 & 5x \\ 0 & 1 \end{pmatrix}.$$

Therefore if $A^5 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $x = 1/5$.

(6) Consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

while

$$BA = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

which are not equal to each other.